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First Edition

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AN

INTRODUCTION

TO

ALGEBRA;

WITH

NOTES AND OBSERVATIONS:

DESIGNED FOR THE USE OF

SCHOOLS, and Places of PUBLIC EDUCATION.

BY JOHN BONNYCASTLE,

Author of the SCHOLAR'S GUIDE TO ARITHMETIC, and a
TREATISE OF MENSURATION.

> ——Ingenuas didiciſſe fideliter artes
> Emollit mores, nec ſinit eſſe feros. OVID.

LONDON:
Printed for J. JOHNSON, No. 72, ST. PAUL'S CHURCH-
YARD, 1782.

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TO THE REV.

J. PRIESTLEY, L.L.D. F.R.S. &c.

SIR,

THE publication of the following treatiſe is owing to your kind encouragement and approbation; and I am happy to embrace this opportunity of teſtifying the high ſenſe I entertain of your condeſcending politeneſs and attention. Whilſt you are enlarging the bounds of ſcience by your numerous and important diſcoveries, you are equally ſolicitous of promoting every other laudable purſuit and uſeful undertaking. And to this amiable diſpoſition the world is no leſs indebted than to your diſtinguiſhed and eminent abilities: the one commands our eſteem and regard, and the other our admiration. Permit me, therefore, Sir, as a ſincere tribute to your merit, to inſcribe to you this compendium, and to aſſure you that

I am,

With the higheſt reſpect,

Your moſt obedient and

obliged humble ſervant,

London,
Sep. 2, 1782.

JOHN BONNYCASTLE.

PREFACE.

THE powers of the mind, like thoſe of the body, are increaſed by continual exertion; application and induſtry ſupply the place of genius and invention; and even the creative faculty itſelf, may be ſtrengthened and improved by uſe and perſeverance. Uncultivated nature is uniformly rude and imbecile; and it is by imitation alone that we at firſt acquire knowledge, and the means of extending its bounds. A juſt and perfect acquaintance with the ſimple elements of ſcience, is a neceſſary ſtep towards our future progreſs and advancement; and this, aſſiſted by laborious inveſtigation and habitual inquiry, will conſtantly lead to eminence and perfection.

Books of rudiments, therefore, conciſely written, well digeſted, and methodically arranged, are treaſures of ineſtimable value; and too many attempts cannot be made to render them perfect and complete. When the firſt principles of any art or ſcience are firmly fixed and rooted in the mind, their application ſoon becomes eaſy, pleaſant, and obvious; the underſtanding is delighted and enlarged; we conceive clearly, reaſon diſtinctly, and form juſt and ſatisfactory concluſions. But, on the contrary, when the mind, inſtead of repoſing on the ſtability of truth, and received principles, is wandering in doubt and uncertainty, our ideas will neceſſarily be confuſed and obſcure; and every ſtep we take, muſt be attended with freſh difficulties and endleſs perplexity.

That the grounds, or fundamental parts, of every ſcience, are dull and unentertaining, is a complaint univerſally made, and a truth not to be denied; but, then, what is obtained with difficulty is remembered with eaſe; and what is purchaſed with pain is poſſeſſed with pleaſure. The ſeeds of knowledge are ſown in every ſoil, but it is by proper culture alone that they are cheriſhed and brought to maturity. A few years of early and aſſiduous application never fails of procuring us the reward of our induſtry; and who is there, that knows the pleaſures and advantages which the ſciences afford, that would think his time miſ-ſpent, or his labours uſeleſs? Riches and honours are the gifts of fortune, caſually beſtowed or hereditarily received, and are frequently abuſed by their poſſeſſors; but the ſuperiority of wiſdom and knowledge is a pre-eminence of merit, that originates with the man, and is the nobleſt of all diſtinctions.

Nature, bountiful and wiſe in all things, has provided us with an infinite variety of ſcenes, both for our inſtruction and entertainment; and, like a kind and indulgent parent, admits all her children to an equal participation of her bleſſings. But as the modes, ſituations, and circumſtances of life are various, ſo accident, habit, and education, have each their predominating influence, and give to every mind its particular bias. Where examples of excellence are wanting, the attempts to attain it are but few; but eminence excites attention, and produces imitation. To raiſe the curioſity, and to awaken the liſtleſs and dormant powers of younger minds, we have only to point out to them a valuable acquiſition, and the means of obtaining it. The active principles are immediately put into motion, and the certainty of the conqueſt is inſured from a determination to conquer.

But of all the ſciences which ſerve to call forth this ſpirit of enterpriſe and inquiry, there are none more eminently uſeful than the Mathematics.

By an early attachment to theſe elegant and ſublime ſtudies, we acquire a habit of reaſoning, and an elevation of thought, that fixes the mind, and prepares it for every other purſuit. From a few ſimple axioms, and evident principles, we proceed gradually to the moſt general propoſitions, and remote analogies; deducing one truth from another, in a chain of argument, well connected and logically purſued; that brings us at laſt, in the moſt ſatisfactory manner, to the concluſion, and ſerves as a general direction in all our inquiries after truth.

And it is not only in this reſpect that mathematical learning is ſo highly valuable: it is, likewiſe, equally eſtimable for its practical utility. Almoſt all the works of art, and devices of man, have a dependence upon its principles, and are indebted to it for their origin and perfection. The cultivation of theſe admirable ſciences is, therefore, a thing of the utmoſt importance, and ought to be conſidered as a principal part of every liberal and well regulated plan of education. They are the guide of our youth, the perfection of our reaſon, and the foundation, or baſis, of every great and noble undertaking.

From theſe conſiderations, I have been induced to undertake an introductory courſe of mathematical ſcience; and, from the kind encouragement I have hitherto received, am not without hopes of a continuance of the ſame candour and approbation. Conſiderable practice as a teacher, and a long attention to the difficulties and obſtructions which retard the progreſs of learners in general, has enabled me to accommodate myſelf the more eaſily to their capacities and underſtandings. And as an earneſt deſire of promoting and diffuſing uſeful knowledge, is the chief motive for this undertaking, ſo no pains, or attention, ſhall be wanting to make it as complete and perfect as poſſible.

The ſubject of the preſent performance is ALGEBRA; which is one of the moſt important and uſeful branches of thoſe ſciences; and may be juſtly conſidered as the key to all the reſt. Geometry delights us by the ſimplicity of its principles, and the elegance of its demonſtrations: Arithmetic is confined in its object, and partial in its application: but Algebra, or the analytic art, is general and comprehenſive, and may be uſed with ſucceſs, in all caſes, where truth is to be obtained, and proper data can be eſtabliſhed.

To trace this noble ſcience to its birth, and to point out all the various alterations and improvements it has undergone in its progreſs, would far exceed the limits of a preface. It will be ſufficient to obſerve, that the invention is of the higheſt antiquity, and has challenged the praiſe and admiration of all ages. *Diophantus* appears to have been the firſt, among the ancients, who applied it to the ſolution of indeterminate and unlimited problems; but it is to the moderns that we are principally indebted for all the moſt curious refinements of the art, and its great and extenſive uſefulneſs in every abſtruſe and difficult inquiry. *Newton*, *Maclaurin*, *Saunderſon*, *Simpſon*, and *Emerſon*, are thoſe, of our own countrymen, who have particularly excelled in this reſpect; and it is to their works that I would refer the young ſtudent, as the patterns of elegance and perfection.

The following compendium is formed entirely upon the model of thoſe writers, and is intended as a uſeful and neceſſary introduction to them. Almoſt every ſubject, that belongs to pure Algebra, is conciſely and diſtinctly treated of; and no pains have been ſpared, to make the whole as eaſy and intelligible as poſſible. A great number of elementary books have already been written upon this ſubject; but there are none, that I have yet ſeen, but what appear to me to be extremely defective. Beſides being totally unfit for the purpoſe of teaching, they are generally calculated to vitiate the taſte, and miſlead the judgment. A tedious and inelegant method prevails through the whole, ſo that the beauty of the ſcience is deſtroyed by the clumſy and aukward manner in which it is treated; and the learner, when he is afterwards introduced to ſome of our beſt writers, is obliged to unlearn and forget every thing that he has been at ſo much pains in acquiring.

It is in the ſciences as in every branch of polite literature; there is a certain taſte and elegance that is only to be obtained from the beſt authors, and a judicious uſe of their inſtructions. To direct the ſtudent in his choice of books, and to prepare him properly for the advantages he may receive from them, is, therefore, the buſineſs of every writer who engages in the humble, but uſeful, taſk of a preliminary tutor. This information I have been careful to give, in every part of the preſent performance, where it could be thought to be in the leaſt uſeful or neceſſary. The nature and confined limits of my plan, admitted not of diffuſe obſervations, or a for enumeration of particalars; but nothing of uſe and importance has been omitted. My p cipal object was to conſult the eaſe, ſatisfacti and accommodation of the learner, and if execution of the work is found equal to the ſign, my purpoſe will be anſwered, and it c not fail of meeting with a candid and favour reception from the public.

ERRATA.

Page 12, line 7, for $xy4$ read $x4-y4$. P. 20, l. 13, $2bx^2$ read $2bx$. P. 22, l. 3, for $a+b$ read $x+b$. P. l. 13, for $\frac{5bx}{a}$ read $\frac{5bx}{2a}$. P. 29, l. 29, dele —. P. 31, l. for $x+b$ read $x+a$. P. 32, l. 10, dele —. P. 49, l. 5, $x5$ read x^6. P. 50, l. 24 and 26, read $a\times^3\surd a$, $3a^3\times^3$ $9a^5\times^3\surd a$, &c. P. 54, l. 33, for + read :: P. 56, l. for 19 read 16. P. 60, l. 24, for $+3x$ read $-3x$, an ult. for ba read $6a$. P. 64, l. 9, for 2 read 3. P. 79, l. dele $\surd$. P, 82, l. 21, for $x-$ read $x=-$, and line ult. negative read affirmative.

ALGEBRA.

DEFINITIONS.

ALGEBRA is the art of computing by ſymbols.

1. *Like quantities* are thoſe that conſiſt of the ſame letters.

2. *Unlike quantities* are thoſe that conſiſt of different letters.

3. *Given quantities* are thoſe whoſe values are known.

4. *Unknown quantities* are thoſe whoſe values are unknown.

5. *Simple quantities* are thoſe that conſiſt of one term only.

6. *Compound quantities* are thoſe that conſiſt of ſeveral terms.

7. *Poſitive* or *affirmative quantities* are thoſe to be added.

8. *Negative quantities* are thoſe to be ſubtracted.

9. *Like ſigns* are all + or all —.

10. *Unlike ſigns* are + and —.

11. *The co-efficient* of any quantity is the n
prefixed to it.

12. *A binomial quantity* is one conſiſting o
terms; a *trinomial* of three terms; and a *quadri*
of four terms; &c.

13. *A reſidual quantity* is a binomial where c
the terms is negative.

14. *The power of a quantity* is its ſquare,
biquadrate &c.

15. *The index* or *exponent* is the number exp
the power to which the quantity is involved.

16. *A rational quantity* is that which has no
cal ſign.

17. *The reciprocal* of any quantity is that qu
inverted, or unity divided thereby.

EXPLANATION OF THE CHARACTERS

$+$ Is the ſign of addition.
$-$ —— of ſubtraction.
$\times$ —— of multiplication.
$\div$ —— of diviſion.
$\surd$ —— of the ſquare root.
$^3\surd$ —— of the cube root.
$^m\surd$ —— of the *m* root.
$=$ —— of equality.

Thus $a+b$ is the ſum of a and b.
$a-b$ is the difference of a and b.
$a \backsim b$ is the difference of a and b whe
not known which is the greate
ab, or $a\times b$, or $a.b$ is the product of a
$a\div b$ or $\frac{a}{b}$ is a divided by b.

$\sqrt{a}$ or $a^{\frac{1}{2}}$ is the ſquare root of a.
$\sqrt[3]{a}$ or $a^{\frac{1}{3}}$ is the cube root of a.
a^2 is the ſquare of a.
a^3 is the cube of a.
a^m is the m power of a.
$a^{\frac{1}{m}}$ is the m root of a.
$\frac{1}{a}$ is the reciprocal of a, and $\frac{a}{b}$ is the reciprocal of $\frac{b}{a}$.

In the computation of problems, the firſt letters of the alphabet are put for known quantities, and the laſt letters for thoſe that are unknown.

AXIOMS.

1. If equal quantities be added to equal quantities the wholes will be equal.

2. If equal quantities be taken from equal quantities, the remainders will be equal.

3. If equal quantities be multiplied by equal quantities, the products will be equal.

4. If equal quantities be divided by equal quantities, the quotients will be equal.

5. The equal powers or roots of equal quantities are equal.

6. Two quantities reſpectively equal to a third, are equal to each other.

7. The whole is equal to all its parts taken together.

ADDITION.

CASE I.

To add quantities that are like, and have like ſigns.

RULE.

Add all the co-efficients together, and to their ſum adjoin the letters common to each term, prefixing the common ſign.

EXAMPLES*:

$$\begin{array}{r}5a\\7a\\8a\\10a\\2a\\a\\\hline 33a\\\hline\end{array}\qquad\begin{array}{r}-6bx\\-3bx\\-2bx\\-7bx\\-bx\\-5bx\\\hline -24bx\\\hline\end{array}\qquad\begin{array}{r}8bxy\\7bxy\\3bxy\\4bxy\\5bxy\\bxy\\\hline 28bxy\\\hline\end{array}$$

$$\begin{array}{r}5x^2+xy\\3x^2+2xy\\x^2+3xy\\7x^2+8xy\\x^2+xy\\\hline 17x^2+15xy\\\hline\end{array}\qquad\begin{array}{r}7ax-y\\8ax-3y\\6ax-2y\\4ax-3y\\ax-y\\\hline 26ax-10y\\\hline\end{array}$$

---

* When a leading quantity has no ſign before it + is always underſtood; and a quantity without any co-efficient prefixed to it is ſuppoſed to have 1, or unity.

$$
\begin{array}{r}
15xy \\ 2xy \\ 7xy \\ xy \\ xy \\ \hline \\ \hline
\end{array}
\qquad
\begin{array}{r}
-8y^2 \\ -7y^2 \\ -y^2 \\ -6y^2 \\ -y^2 \\ \hline \\ \hline
\end{array}
\qquad
\begin{array}{r}
7a-6b \\ 4a-3b \\ 2a-8b \\ a-b \\ 3a-2b \\ \hline \\ \hline
\end{array}
$$

$$
\begin{array}{r}
3x^{\frac{1}{2}}-xy \\ 2x^{\frac{1}{2}}-3xy \\ 4x^{\frac{1}{2}}-8xy \\ x^{\frac{1}{2}}-2xy \\ \hline \\ \hline
\end{array}
\qquad
\begin{array}{r}
3xy-x+2ab \\ 2xy-3x+2ab \\ 2xy-4x+8ab \\ 5xy-3x+ab \\ \hline \\ \hline
\end{array}
$$

CASE II.

To add quantities that are like, but have unlike ſigns.

RULE,

1. Add all the affirmative co-efficients into one ſum, and all the negative ones into another.

2. Subtract the leaſt ſum from the greateſt, and to the difference prefix the ſign of the greateſt, with the common quantity.

EXAMPLES:

$$\begin{array}{r} -3a \\ +7a \\ +8a \\ -\ a \\ -2a \\ \hline +9a \\ \hline \end{array} \qquad \begin{array}{r} +8ax^2 \\ +7ax^2 \\ -3ax^2 \\ -4ax^2 \\ +4ax^2 \\ \hline +12ax^2 \\ \hline \end{array} \qquad \begin{array}{r} +6x^{\frac{1}{3}}+8y \\ -3x^{\frac{1}{3}}+7y \\ -13x^{\frac{1}{3}}+8y \\ +2x^{\frac{1}{3}}-3y \\ +x^{\frac{1}{3}}-y \\ \hline -7x^{\frac{1}{3}}+19y \\ \hline \end{array}$$

$$\begin{array}{r} -2xy+8 \\ -3xy+7 \\ +xy-10 \\ +5xy-7 \\ \hline +xy-2 \\ \hline \end{array} \qquad \begin{array}{r} +8x^2-y+3\sqrt{x} \\ -10x^2-3y+2\sqrt{x} \\ -4x^2-2y+\sqrt{x} \\ +9x^2+6y-10\sqrt{x} \\ \hline +3x^2 \cdot\cdot * -4\sqrt{x} \\ \hline \end{array}$$

$$\begin{array}{r} -2a^2 \\ -3a^2 \\ -8a^2 \\ +10a^2 \\ +13a^2 \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} +8b^2y^3 \\ +6b^2y^3 \\ -10b^2y^3 \\ -20b^2y^3 \\ -b^2y^3 \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} -3ab+3 \\ +8ab-10 \\ +3ab-6 \\ -ab+2 \\ -2ab+11 \\ \hline \\ \hline \end{array}$$

$$\begin{array}{r} +4\sqrt{bx}-x \\ -3\sqrt{bx}-3x \\ -\sqrt{bx}-8x \\ -6\sqrt{bx}+11x \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} +4x^2-y^2-2 \\ +10x^2-8y^2-4 \\ -6x^2-y^2+8 \\ +3x^2+10y^2-1 \\ \hline \\ \hline \end{array}$$

CASE III.

To add quantities that are unlike, and have unlike ſigns.

RULE.

Collect the like quantities together by the laſt rule, and ſet down thoſe that are unlike, one after another, with their proper ſigns.

EXAMPLES:

$$\begin{array}{r} 2x \\ 3y \\ -a \\ x^2 \\ \hline 2x+3y-a+x^2 \\ \hline \end{array} \qquad \begin{array}{l} 2x-x^2 \\ 3a-x \\ 2ax+6x^2 \\ 3\sqrt{x}-2ax \\ \hline x+5x^2+3a+3\sqrt{x} \\ \hline \end{array}$$

$$\begin{array}{r} 12ax-x^2-6+\sqrt{ax-x^2} \\ -6ax+x^2-x+10 \\ 3y-ax-4-2\sqrt{ax-x^2} \\ \hline 5ax-\sqrt{ax-x^2}-x+3y \\ \hline \end{array}$$

$$\begin{array}{r} 3x^2y \\ -2xy \\ -3y^2x \\ -8x^2y \\ \hline \\ \hline \end{array} \qquad \begin{array}{l} 2\sqrt{x}-8 \\ 3\sqrt{xy}+10 \\ 6ab+x \\ -8+\sqrt{xy} \\ \hline \\ \hline \end{array} \qquad \begin{array}{l} 3a^2-8+x^{\frac{1}{2}}-2a \\ 3a-10+a^2-6a \\ x^2-4a^2+8-4\sqrt{x} \\ 10-a-x^2-x^{\frac{1}{3}} \\ \hline \\ \hline \end{array}$$

SUBTRACTION.

RULE.

Change the ſigns of all the quantities to be ſubtracted, and then add them together as in addition, and the reſult will be the remainder required.

EXAMPLES:

$$\begin{array}{l} 3a^2-2b \\ 2a^2-3b \\ \hline a^2+b \\ \hline \end{array} \qquad \begin{array}{l} 6x^2-8y+2 \\ x^2+9y-20 \\ \hline 5x^2-17y+22 \\ \hline \end{array} \qquad \begin{array}{l} 35xy-2+8x-y^{\frac{1}{2}} \\ 24xy-8-8x-3y \\ \hline 11xy+6+16x-y^{\frac{1}{2}}+3y \\ \hline \end{array}$$

$$\begin{array}{l} 8ax-2\sqrt{xy}-10 \\ 10x-6\sqrt{xy}-ax \\ \hline 9ax+4\sqrt{xy}-10-10x \\ \hline \end{array} \qquad \begin{array}{l} 4\sqrt{\frac{x}{a}}-10-8x-3xy \\ -5xy-7x+3-y \\ \hline 4\sqrt{\frac{x}{a}}-13-x+2xy+y \\ \hline \end{array}$$

$$\begin{array}{l} 5x^2y-8 \\ -3x^2y+10 \\ \hline \\ \hline \end{array} \qquad \begin{array}{l} 4\sqrt{xy}-x\sqrt{xy} \\ 2\sqrt{xy}+12+xy \\ \hline \\ \hline \end{array} \qquad \begin{array}{l} 5x^2y^2+\sqrt{\frac{x}{a}}-8-4b \\ 6x^2y^2-10+4b-\sqrt{\frac{x}{2}} \\ \hline \\ \hline \end{array}$$

MULTIPLICATION.

CASE I.

To multiply ſimple quantities.

RULE.

Multiply the co-efficients of the two terms together, and to the product prefix all the letters in thoſe terms, and the reſult will be the whole product required.

Note. Like ſigns produce +, and unlike ſigns −.

EXAMPLES:

$$\begin{array}{cccc} 2a & -2a & 5a & -9x \\ 3b & +4b & -6x & -5b \\ \hline 6ab & -8ab & -30ax & +45bx \\ \hline \end{array}$$

$$\begin{array}{cccc} 7ab & 6a^2x & -x^2y & -7xy \\ -5ac & 5x & xy^2 & -xy \\ \hline -35a^2bc & 30a^2x^2 & -x^3y^3 & +7x^2y^2 \\ \hline \end{array}$$

$$\begin{array}{cccc} a & -ax & +5xy & -xy \\ x & -7b & -3 & +6ax \\ \hline & & & \\ \hline \end{array}$$

CASE II.

When one of the factors is a compound quantit

RULE.

Find the products of the multiplier, and every ticular term of the multiplicand ſeparately, place them one after another, with their proper ſi and the reſult will be the whole product require

EXAMPLES:

$$\begin{array}{l} 4a-2b \\ 3a \\ \hline 12a^2-6ab \\ \hline \end{array} \qquad \begin{array}{l} 6xy-8 \\ 2x \\ \hline 12x^2y-16x \\ \hline \end{array} \qquad \begin{array}{l} 8a^2-2x+6 \\ 3xy \\ \hline 24a^2xy-6x^2y+18 \\ \hline \end{array}$$

$$\begin{array}{l} 3x-a \\ 2a \\ \hline \\ \hline \end{array} \qquad \begin{array}{l} 5x-7a \\ -x \\ \hline \\ \hline \end{array} \qquad \begin{array}{l} 3y-8+2xy \\ xy \\ \hline \\ \hline \end{array}$$

CASE III.

When both the factors are compound quantitie

RULE.

Multiply every particular term of the multi into every term of the multiplicand reſpectively ſet down the products one after another with proper ſigns, and their ſum will be the whole duct required.

EXAMPLES:

$$\begin{array}{l} x+y \\ x+y \\ \hline x^2+xy \\ \quad +xy+y^2 \\ \hline x^2+2xy+y^2 \\ \hline \end{array} \qquad \begin{array}{l} x+y \\ x-y \\ \hline x^2+xy \\ \quad -xy-y^2 \\ \hline x^2 \quad * \quad -y^2 \\ \hline \end{array} \qquad \begin{array}{l} x-y \\ x-y \\ \hline x^2-xy \\ \quad -xy+y^2 \\ \hline x^2-2xy+y^2 \\ \hline \end{array}$$

$$\begin{array}{l} 5x+4y \\ 3x-2y \\ \hline 15x^2+12xy \\ \quad -10xy-8y^2 \\ \hline 15x^2+2xy-8y^2 \\ \hline \end{array} \qquad \begin{array}{l} x^2+xy-y^2 \\ x-y \\ \hline x^3+x^2y-xy^2 \\ \quad -x^2y-xy^2-y^3 \\ \hline x^3 \quad * \quad 2xy^2-y^3 \\ \hline \end{array}$$

$$\begin{array}{l} x^2+xy+y^2 \\ x-y \\ \hline x^3+x^2y+xy^2 \\ \quad -x^2y-xy^2-y^3 \\ \hline x^3 \quad * \quad * \quad -y^3 \\ \hline \end{array} \qquad \begin{array}{l} x+y-z \\ x-y+z \\ \hline x^2+xy-xz \\ \quad -xy-y^2+yz \\ \qquad\quad +xz+yz-z^2 \\ \hline x^2 \quad * \quad -y^2+2yz-z^2 \\ \hline \end{array}$$

$$\begin{array}{l} 3x^2-2xy+5 \\ \quad x^2+2xy-3 \\ \hline 3x^4-2x^3y+5x^2 \\ \quad +6x^3y-4x^2y^2+10xy \\ \qquad\qquad -9x^2+6xy-15 \\ \hline 3x^4+4x^3y-4x^2-4x^2y^2+16xy-15 \\ \hline \end{array}$$

EXAMPLES FOR PRACTICE.

1. Multiply $12ax$ by $3a$. *Product* $36a^2x$
2. Multiply $4x^2-2y$ by $2y$. *Product* $8x^2y-4y^2$
3. Multiply $2x+4y$ by $2x-4y$. *Product* $4x^2-16y^2$
4. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. *Product* $x \quad y^4$
5. Multiply x^2+xy+y^2 by x^2-xy+y^2. *Product* $x^4+x^2y^2+y^4$
6. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$. *Product* $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$

DIVISION.

CASE I.

When the diviſor is a ſimple quantity.

RULE.

1. Place the dividend above a ſmall line, and the diviſor under it, in the manner of a vulgar fraction.

2. Expunge thoſe letters that are common to both the dividend and diviſor, and divide the co-efficients of all the terms by any number that will divide them without a remainder, and the reſult will be the quotient required.

Note. Like ſigns make +, and unlike ſigns —, the ſame as in multiplication.

EXAMPLES:

$$\frac{a}{a}=1;\quad \frac{8bc}{2b}=4c;\quad \frac{abc}{bcd}=\frac{a}{d};\quad \frac{10ab+15ac}{20ad}=\frac{2b+3c}{4d};$$

$$\frac{ab+b^2}{2b}=\frac{a+b}{2};\quad \frac{12xy}{6x^2}=\frac{2y}{x};\quad \frac{30ax-54ay}{12ab}=\frac{5x-9y}{2b};$$

$$\frac{10x^2y-15y^2-5y}{5y}=2x^2-3y-1.$$

1. Divide $18x^2$ by $9x$. *Quotient* $2x$.
2. Divide $10x^2y^2$ by $-5x^2y$. *Quotient* $-2y$
3. Divide $-9ax^2y^2$ by $9x^2y$. *Quotient* $-ay$
4. Divide $-8x^2$ by $-2x$. *Quotient* $+4x$
5. Divide $a+3ax-x^2$ by a. *Quotient* $1+3x-\frac{x^2}{a}$
6. Divide $3a^2-15+6a+3b$ by $3a$.

Quotient $a-\frac{5}{a}+2+\frac{b}{a}$.

CASE II.

When the diviſor and dividend are both compound quantities.

RULE.

1. Range the terms of both the quantities according to the dimenſions of ſome letter in them; ſo that the firſt term may have the higheſt power of that letter, and the ſecond term the next higheſt power; and ſo on.

2. Divide the firſt term of the dividend by the firſt term of the diviſor, and place the reſult in the quotient.

EXAMPLES FOR PRACTICE.

1. Multiply $12ax$ by $3a$. *Product* $36a^2x$
2. Multiply $4x^2-2y$ by $2y$. *Product* $8x^2y-4y^2$
3. Multiply $2x+4y$ by $2x-4y$. *Product* $4x^2-16y^2$
4. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. *Product* $x \quad y^4$
5. Multiply x^2+xy+y^2 by x^2-xy+y^2. *Product* $x^4+x^2y^2+y^4$
6. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$. *Product* $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$

DIVISION.

CASE I.

When the diviſor is a ſimple quantity.

RULE.

1. Place the dividend above a ſmall line, and the diviſor under it, in the manner of a vulgar fraction.

2. Expunge thoſe letters that are common to both the dividend and diviſor, and divide the co-efficients of all the terms by any number that will divide them without a remainder, and the reſult will be the quotient required.

Note. Like ſigns make +, and unlike ſigns —, the ſame as in multiplication.

EXAMPLES:

$$\frac{a}{a}=1;\ \frac{8bc}{2b}=4c;\ \frac{abc}{bcd}=\frac{a}{d};\ \frac{10ab+15ac}{20ad}=\frac{2b+3c}{4d};$$

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5. Divide $a+3ax-x^2$ by a. *Quotient* $1+3x-\frac{x^2}{a}$
6. Divide $3a^2-15+6a+3b$ by $3a$.

Quotient $a-\frac{5}{a}+2+\frac{b}{a}$.

CASE II.

When the diviſor and dividend are both compound quantities.

RULE.

1. Range the terms of both the quantities according to the dimenſions of ſome letter in them; ſo that the firſt term may have the higheſt power of that letter, and the ſecond term the next higheſt power; and ſo on.

2. Divide the firſt term of the dividend by the firſt term of the diviſor, and place the reſult in the quotient.

EXAMPLES FOR PRACTICE.

1. Multiply $12ax$ by $3a$. *Product* $36a^2x$
2. Multiply $4x^2-2y$ by $2y$. *Product* $8x^2y-4y^2$
3. Multiply $2x+4y$ by $2x-4y$. *Product* $4x^2-16y^2$
4. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. *Product* $x \quad y^4$
5. Multiply x^2+xy+y^2 by x^2-xy+y^2. *Product* $x^4+x^2y^2+y^4$
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DIVISION.

CASE I.

When the diviſor is a ſimple quantity.

RULE.

1. Place the dividend above a ſmall line, and the diviſor under it, in the manner of a vulgar fraction.

2. Expunge thoſe letters that are common to both the dividend and diviſor, and divide the co-efficients of all the terms by any number that will divide them without a remainder, and the reſult will be the quotient required.

Note. Like ſigns make +, and unlike ſigns —, the ſame as in multiplication.

EXAMPLES:

$\frac{a}{a}=1$; $\frac{8bc}{2b}=4c$; $\frac{abc}{bcd}=\frac{a}{d}$; $\frac{10ab+15ac}{20ad}=\frac{2b+3c}{4d}$;

$\frac{ab+b^2}{2b}=\frac{a+b}{2}$; $\frac{12xy}{6x^2}=\frac{2y}{x}$; $\frac{30ax-54ay}{12ab}=\frac{5x-9y}{2b}$;

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4. Divide $-8x^2$ by $-2x$. *Quotient* $+4x$
5. Divide $a+3ax-x^2$ by a. *Quotient* $1+3x-\frac{x^2}{a}$
6. Divide $3a^2-15+6a+3b$ by $3a$.

Quotient $a-\frac{5}{a}+2+\frac{b}{a}$.

CASE II.

When the diviſor and dividend are both compound quantities.

RULE.

1. Range the terms of both the quantities according to the dimenſions of ſome letter in them; ſo that the firſt term may have the higheſt power of that letter, and the ſecond term the next higheſt power; and ſo on.

2. Divide the firſt term of the dividend by the firſt term of the diviſor, and place the reſult in the quotient.

EXAMPLES FOR PRACTICE.

1. Multiply $12ax$ by $3a$. *Product* $36a^2x$
2. Multiply $4x^2-2y$ by $2y$. *Product* $8x^2y-4y^2$
3. Multiply $2x+4y$ by $2x-4y$. *Product* $4x^2-16y^2$
4. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. *Product* $x \quad y^4$
5. Multiply x^2+xy+y^2 by x^2-xy+y^2. *Product* $x^4+x^2y^2+y^4$
6. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$. *Product* $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$

DIVISION.

CASE I.

When the diviſor is a ſimple quantity.

RULE.

1. Place the dividend above a ſmall line, and the diviſor under it, in the manner of a vulgar fraction.

2. Expunge thoſe letters that are common to both the dividend and diviſor, and divide the co-efficients of all the terms by any number that will divide them without a remainder, and the reſult will be the quotient required.

Note. Like ſigns make +, and unlike ſigns —, the ſame as in multiplication.

EXAMPLES:

$$\frac{a}{a}=1;\quad \frac{8bc}{2b}=4c;\quad \frac{abc}{bcd}=\frac{a}{d};\quad \frac{10ab+15ac}{20ad}=\frac{2b+3c}{4d};$$

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4. Divide $-8x^2$ by $-2x$. *Quotient* $+4x$
5. Divide $a+3ax-x^2$ by a. *Quotient* $1+3x-\frac{x^2}{a}$
6. Divide $3a^2-15+6a+3b$ by $3a$.

   Quotient $a-\frac{5}{a}+2+\frac{b}{a}$.

CASE II.

When the diviſor and dividend are both compound quantities.

RULE.

1. Range the terms of both the quantities according to the dimenſions of ſome letter in them; ſo that the firſt term may have the higheſt power of that letter, and the ſecond term the next higheſt power; and ſo on.

2. Divide the firſt term of the dividend by the firſt term of the diviſor, and place the reſult in the quotient.

3. Multiply the whole diviſor by the quotient term laſt found, and ſubtract the reſult from the dividend.

4. To this remainder bring down the next term of the dividend, and divide as before; and ſo on, as in common arithmetic.

Note, Like ſigns produce +, and unlike ſigns —.

EXAMPLES:

$$\begin{array}{l}
a+x)\,a^3+5a^2x+5ax^2+x^3\,(a^2+4ax+x^2 \\
\qquad\underline{a^3+a^2x} \\
\qquad\qquad 4a^2x+5ax^2 \\
\qquad\qquad \underline{4a^2x+4ax^2} \\
\qquad\qquad\qquad\qquad ax^2+x^3 \\
\qquad\qquad\qquad\qquad \underline{ax^2+x^3} \\
\qquad\qquad\qquad\qquad\qquad *
\end{array}$$

$$\begin{array}{l}
x-3)\,x^3-9x^2+27x-27\,(x^2-6x+9 \\
\qquad\underline{x^3-3x^2} \\
\qquad\qquad -6x^2+27x \\
\qquad\qquad \underline{-6x^2+18x} \\
\qquad\qquad\qquad\qquad 9x-27 \\
\qquad\qquad\qquad\qquad \underline{9x-27} \\
\qquad\qquad\qquad\qquad\quad *
\end{array}$$

$$\begin{array}{lllll}
a-x) & a^3-x^3\,(a^2+ax+x^2 & & \\
 & a^3-a^2x & & \\
\cline{2-2}
 & & a^2x-x^3 & \\
 & & a^2x-ax^2 & \\
\cline{3-3}
 & & & ax^2-x^3 \\
 & & & ax^2-x^3 \\
\cline{4-4}
 & & & *
\end{array}$$

$$\begin{array}{llllll}
b-y) & b^4-y^4\,(b^3+b^2y+by^2+y^3 & & & \\
 & b^4-b^3y & & & \\
\cline{2-2}
 & & b^3y-y^4 & & \\
 & & b^3y-b^2y^2 & & \\
\cline{3-3}
 & & & b^2y^2-y^4 & \\
 & & & b^2y^2-by^3 & \\
\cline{4-4}
 & & & & by^3-y^4 \\
 & & & & by^3-y^4 \\
\cline{5-5}
 & & & & *
\end{array}$$

EXAMPLES FOR PRACTICE.

1. Divide $a^2+2ax+x^2$ by $a+x$. *Quotient* $a+x$
2. Divide $a^3-3a^2y+3ay^2-y^3$ by $a-y$. *Quotient* $a^2-2ay+y^2$
3. Divide 1 by $1-x$. *Quotient* $1+x+x^2+x^3$ &c.
4. Divide $6x^4-96$ by $3x-6$. *Quotient* $2x^3+4x^2+8x+16$
5. Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$. *Quotient* $a^3-3a^2x+3ax^2-x^3$

ALGEBRAIC FRACTIONS.

PROBLEM I.

To reduce a mixed quantity to an improper fraction.

RULE.

Multiply the integer by the denominator of the fraction, and to the product add the numerator; and the denominator being placed under this ſum will give the improper fraction required.

EXAMPLES:

$$3\frac{5}{7}=\frac{3\times7+5}{7}=\frac{26}{7};\qquad x+\frac{x^2}{a}=\frac{ax+x^2}{a};$$

$$a-\frac{b}{c}=\frac{ac-b}{c};\qquad 5\frac{2}{3}=\frac{5\times3+2}{3}=\frac{17}{3};$$

$$1-\frac{2x}{a}=\frac{a-2x}{a};\qquad a-x+\frac{a^2-ax}{x}=\frac{a^2-x^2}{x}.$$

Let the following mixed quantities be reduced to improper fractions:

$$8\frac{6}{7};\quad a^2-\frac{2a}{b};\quad 1+2x-\frac{x-3}{5x};\quad b+\frac{1-x+a}{c};$$

PROBLEM II.

To reduce an improper fraction to a whole or mixed quantity.

RULE.

Divide the numerator by the denominator for the integral part, and place the remainder over the denominator for the fractional part, and it will be the mixed quantity required.

EXAMPLES.

$$\frac{17}{5}=3\frac{2}{5};\ \frac{ax+a^2}{x}=a+\frac{a^2}{x};\ \frac{ay+2y^2}{a+y}=y+\frac{y^2}{a+y};$$

$$\frac{25}{4}=6\frac{1}{4};\ \frac{ab-a^2}{b}=a-\frac{a^2}{b};\ \frac{a^2+x^2}{a-x}=a+x+\frac{2x^2}{a-x}.$$

Let the following improper fractions be reduced to mixed quantities.

$$\frac{35}{8};\ \frac{28}{4};\ \frac{3ab-b^2}{a};\ \frac{2x^2y}{2x};\ \frac{b-2x^2}{b+x};$$

PROBLEM III.

To reduce fractions of different denominators, to thoſe of the ſame value that ſhall have a common denominator.

RULE.

Multiply every numerator ſeparately into all the denominators but its own for the new numerators, and all the denominators together for the common denominator required.

EXAMPLES:

1. Reduce $\frac{a}{b}$ *and* $\frac{b}{c}$ to fractions of equal values that ſhall have a common denominator.

$$a \times c = ac$$
$$b \times b = b^2$$
$$b \times c = bc$$

$\frac{ac}{bc}$ *and* $\frac{b^2}{bc}$ = *fractions required.*

2. Reduce $\frac{a}{b}$, $\frac{b}{c}$ *and* $\frac{c}{d}$ to equivalent fractions, having a common denominator.

$$a \times c \times d = acd$$
$$b \times b \times d = b^2d$$
$$c \times b \times c = c^2b$$
$$b \times c \times d = bcd$$

$\frac{acd}{bcd}$, $\frac{b^2d}{bcd}$, *and* $\frac{c^2b}{bcd}$ = *fractions required.*

3. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to equivalent fractions, having a common denominator. *Anſ.* $\frac{2xc}{ac}$ *and* $\frac{ab}{ac}$.

4. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to a common denominator. *Anſ.* $\frac{ac}{bc}$ *and* $\frac{ab+b^2}{bc}$.

5. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$ and d to a common denominator. *Anſ.* $\frac{9xc}{6ac}$, $\frac{4ab}{6ac}$, *and* $\frac{6adc}{6ac}$.

6. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a+\frac{2x}{a}$ to a common denominator. *Anſ.* $\frac{9a}{12a}$, $\frac{8ax}{12a}$ *and* $\frac{12a^2+24x}{12a}$.

PROBLEM IV.

To find the greateſt common meaſure of a fraction.

RULE.

1. Range the quantities according to the dimenſions of ſome letter, as is ſhewn in diviſion.

2. Divide the greater term by the leſs, and the laſt diviſor by the laſt remainder, and ſo on till nothing remains; and the diviſor laſt uſed will be the common meaſure required.

Note, All the letters or figures that are common to each diviſor, muſt be thrown out of them before they are uſed in the operation.

EXAMPLES:

1. To find the greateſt common meaſure of $\frac{cx+x^2}{ca^2+a^2x}$.

$$\begin{array}{rl}
cx+x^2) & ca^2+a^2x \\
\text{or } c+x\ ) & ca^2+a^2x(a^2 \\
 & \underline{ca^2+a^2x} \\
 & \quad *
\end{array}$$

Therefore the greateſt common meaſure is $c+x$.

2. To find the greateſt common meaſure of $\frac{x^3-b^2x}{x^2+2bx^2+b^2}$.

$$\begin{array}{rl}
x^2+2bx+b^2) & x^3-b^2x(x \\
 & \underline{x^3+2bx^2+b^2x} \\
-2bx^2-2b^2x) & x^2+2bx+b^2 \\
\text{or } \quad x+b \quad) & x^2+2bx+b^2(x+b \\
 & \underline{x^2+bx} \\
 & \qquad bx+b^2 \\
 & \qquad \underline{bx+b^2} \\
 & \qquad\quad *
\end{array}$$

Therefore $x+b$ is the greateſt common diviſor.

3. To find the greateſt common diviſor of $\frac{x^4-b^4}{x^5-b^2x^3}$. *Anſ.* x^2-b^2.

4. To find the greateſt common meaſure of $\frac{5a^5+10a^4b+5a^3b^2}{a^3b+2a^2b^2+2ab^3+b^4}$. *Anſ.* $a+b$.

PROBLEM V.

To reduce a fraction to its loweſt terms.

RULE.

1. Find the greateſt common meaſure, as in the laſt problem.

2. Divide both the terms of the fraction by the common meaſure thus found, and it will reduce the fraction as required.

EXAMPLES:

1. Reduce $\frac{cx+x^2}{ca^2+a^2x}$ to its loweſt terms.

$$\begin{array}{r} cx+x^2)ca^2+a^2x \\ \text{or } c+x\;)ca^2+a^2x(a^2 \\ ca^2+a^2x \\ \hline * \end{array}$$

Therefore $c+x$ *is the greateſt common meaſure,* and $c+x)\frac{cx+x^2}{ca^2+a^2x}=\frac{x}{a^2}=$ *fraction required.*

2. Having $\frac{x^3-b^2x}{x^2+2bx+b^2}$ given, it is required to reduce it to its leaſt terms.

$$\begin{array}{r} x^2+2bx+b^2)x^3-b^2x(x \\ x^3+2bx^2+b^2x \\ \hline -2bx^2-2b^2x)x^2+2bx+b^2 \\ \text{or } x+b)x^2+2bx+b^2(x+b \\ x^2+bx \\ \hline bx+b^2 \\ bx+b^2 \\ \hline * \end{array}$$

Therefore $x+b$ *is the greateſt common meaſure,*

and $x+b)\dfrac{x^3-b^2x}{x^2+2bx+b^2}=\dfrac{x^2-bx}{a+b}=$ *fraction required.*

3. To reduce $\dfrac{x^4-b^4}{x^5-b^2x^3}$ to its loweſt terms.

Anſ. $\dfrac{x^2+b^2}{x^3}$.

4. Reduce $\dfrac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its loweſt terms.

Anſ. $\dfrac{a^2+x^2}{a-x}$.

PROBLEM VI.

To add fractional quantities together.

RULE.

1. Reduce the fractions to a common denominator, as in problem the third.

2. Add all the numerators together, and under their ſum write the common denominator, and it will give the ſum of the fractions required.

EXAMPLES.

1. Having $\dfrac{x}{2}$ *and* $\dfrac{x}{3}$ given, to find their ſum.

$$\begin{array}{l} x\times 3 = 3x \\ x\times 2 = 2x \\ \hline 2\times 3 = 6 \end{array}$$

$\dfrac{3x}{6}+\dfrac{2x}{6}=\dfrac{5x}{6}=$ *ſum required.*

2. Having $\frac{a}{b}$, $\frac{c}{d}$ *and* $\frac{e}{f}$ given, to find their ſum.

$$\begin{array}{l} a\times d\times f = adf \\ c\times b\times f = cbf \\ e\times b\times d = ebd \\ \hline b\times d\times f = bdf \end{array}$$

$$\frac{adf}{bdf}+\frac{cbf}{bdf}+\frac{ebd}{bdf}=\frac{adf+cbf+ebd}{bdf}= \textit{ſum required.}$$

3. Add $a-\frac{3x^2}{b}$ *and* $b+\frac{x-2b}{c}$ together.

$$\begin{array}{l} 3x^2\times c = 3cx^2 \\ \overline{x-2b}\times b = bx-2b^2 \\ \hline b\times c = bc \end{array}$$

$$\begin{array}{l} a - \frac{3cx^2}{bc} \\ b + \frac{bx-2b^2}{bc} \\ \hline a+b+\frac{bx-3cx^2-2b^2}{bc} = \textit{ſum required.} \end{array}$$

4. Add $\frac{3x}{2b}$ *and* $\frac{x}{5}$ together. *Sum* $\frac{15x+2bx}{10b}$

5. Add $\frac{x}{2}$, $\frac{x}{3}$ *and* $\frac{x}{4}$ together. *Sum* $x+\frac{x}{12}$

6. Add $\frac{x-2}{3}$ *and* $\frac{4x}{7}$ together. *Sum* $\frac{19x-14}{21}$

7. Add $x+\frac{x-2}{3}$ *and* $3x+\frac{2x-3}{4}$ together.

Sum. $4x+\frac{10x-17}{12}$

PROBLEM VII.

To ſubtract one fractional quantity from another.

RULE.

1. Reduce the fractions to a common denominator, as in addition.

2. Subtract the numerators from each other, and under their difference write the common denominator, and it will give the difference of the fractions required.

EXAMPLES:

1. To find the difference of $\frac{x}{3}$ and $\frac{2x}{11}$.

$$x\times11=11x$$
$$2x\times3=6x$$

$$3\times11=33$$

$$\frac{11x}{33}-\frac{6x}{33}=\frac{5x}{33}=$$ *difference required.*

2. To find the difference of $\frac{x-a}{3b}$ and $\frac{2a-4x}{5c}$.

$$\overline{x-a}\times5c=5cx-5ac$$
$$\overline{2a-4x}\times3b=6ab-12bx$$

$$3b\times5c=15bc$$

$$\frac{5cx-5ac}{15bc}-\frac{6ab-12bx}{15bc}=\frac{5cx-5ac-6ab+12bx}{15bc}=$$ *difference required.*

3. From $\frac{3x}{7}$ take $\frac{2x}{9}$. *Difference* $\frac{13x}{63}$.

4. From $\frac{x+a}{b}$ take $\frac{c}{d}$. *Difference* $\frac{dx+ad-bc}{bd}$.

5. From $\frac{3x+a}{5b}$ take $\frac{2x+7}{8}$.

Difference $\frac{24x+8a-10bx-35b}{40b}$.

6. From $3x+\frac{x^2}{b}$ take $x-\frac{x-a}{c}$.

Difference $2x+\frac{cx^2-bx+ab}{bc}$.

PROBLEM VIII.

To multiply fractional quantities together.

RULE.

Multiply the numerators together for a new numerator, and the denominators for a new denominator, and it will give the product required.

EXAMPLES:

1. Find the product of $\frac{x}{6}$ and $\frac{2x}{9}$.

$$\left.\begin{matrix} x\times 2x \\ 6\times 9 \end{matrix}\right\}=\frac{2x^2}{54}=\frac{x^2}{27}= \textit{product required.}$$

2. Find the product of $\frac{x}{2}$, $\frac{4x}{5}$ and $\frac{10x}{21}$.

$$\left.\begin{matrix} x\times4x\times10x \\ 2\times5\times21\end{matrix}\right\}=\frac{40x^3}{210}=\frac{4x^3}{21}= \textit{product requ}$$

3. Find the product of $\frac{x}{a}$ and $\frac{x+a}{a+c}$.

$$\left.\begin{matrix} x\times\overline{x+a} \\ a\times\overline{a+c}\end{matrix}\right\}=\frac{x^2+ax}{a^2+ac}= \textit{product requ}$$

4. Find the product of $\frac{3x}{2}$ and $\frac{3a}{b}$. *Product*

5. Find the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$. *Product* $\frac{3}{5}$

6. Find the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$, $\frac{3ac}{2b}$. *Product*

7. Find the product of $b+\frac{bx}{a}$ and $\frac{a}{x}$.

Product $\frac{ab+}{x}$

8. Find the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$.

Product $\frac{x^4-}{cb^2-}$

PROBLEM IX.

To divide one fractional quantity by another.

RULE.

Invert the diviſor, and proceed as in multip tion.

EXAMPLES:

1. Find the quotient of $\frac{x}{3}$ divided by $\frac{2x}{9}$.

$$\frac{x}{3}\times\frac{9}{2x}=\frac{9x}{6x}=\frac{3}{2}=1\tfrac{1}{2}=\textit{quotient required.}$$

2. Find the quotient of $\frac{2a}{b}$ divided by $\frac{4c}{d}$.

$$\frac{2a}{b}\times\frac{d}{4c}=\frac{2ad}{4bc}=\frac{ad}{2bc}=\textit{quotient required.}$$

3. Find the quotient of $\frac{x+a}{2x-2b}$ divided by $\frac{x+b}{5x+a}$.

$$\frac{x+a}{2x-2b}\times\frac{5x+a}{x+b}=\frac{5x^2+6ax+a^2}{2x^2-2b^2}=\textit{quotient requir.}$$

4. Divide $\frac{5x}{3}$ by $\frac{2a}{3b}$. *Quotient* $\frac{5bx}{a}$.

5. Divide $\frac{x-b}{8cd}$ by $\frac{3cx}{4d}$. *Quotient* $\frac{x-b}{6c^2x}$.

6. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$.

Quotient $x+\frac{b^2}{x}$.

INVOLUTION.

Involution is the raiſing of powers from any propoſed root; or the method of finding the ſquare, cube, biquadrate, &c. of any given quantity.

RULE.

Multiply the quantity into itſelf as often as is noted by the index, and the laſt product will be power required. *Or*

Multiply the index of the quantity by the i of the power, and the reſult will be the ſam before.

Note, When the ſign of the root is + all powers of it will be +; and when the ſign i all the odd powers will be —, and all the powers +.

EXAMPLES:

$$a\text{, root}\left\{\begin{array}{l} a^2 = \textit{ſquare} \\ a^3 = \textit{cube} \\ a^4 = \textit{4th power} \\ a^5 = \textit{5th power} \\ \quad \&c. \end{array}\right. \qquad a^2\text{, root}\left\{\begin{array}{l} a^4 = \textit{ſquare} \\ a^6 = \textit{cube} \\ a^8 = \textit{4th po} \\ a^{10} = \textit{5th po} \\ \quad \&c. \end{array}\right.$$

$$-3a\text{, root}\left\{\begin{array}{l} +\ 9a^2 = \textit{ſquare} \\ -\ 27a^3 = \textit{cube} \\ +\ 81a^4 = \textit{4th power} \\ -243a^5 = \textit{5th power} \end{array}\right.$$

$$-2ax^2\text{, root}\left\{\begin{array}{l} +\ 4a^2x^4 = \textit{ſquare} \\ -\ 8a^3x^6 = \textit{cube} \\ +16a^4x^8 = \textit{4th power} \\ -32a^5x^{10} = \textit{5th power} \end{array}\right.$$

$$\frac{x}{a}\text{, root}\left\{\begin{array}{l} \dfrac{x^2}{a^2} = \textit{ſquare} \\ \dfrac{x^3}{a^3} = \textit{cube} \\ \dfrac{x^4}{a^4} = \textit{biquadrate} \end{array}\right.$$

$$-\frac{2ax^2}{3b},\ \textit{root}\ \left.\vphantom{\begin{array}{l}1\\1\\1\end{array}}\right\}\begin{array}{l} +\dfrac{4a^2x^4}{9b^2}=\textit{ſquare}\\ -\dfrac{8a^3x^6}{27b^3}=\textit{cube}\\ +\dfrac{16a^4x^8}{81b^4}=\textit{4th power}\end{array}$$

$$\begin{array}{lllll}
x+a=\textit{root} & & & & \\
x+a & & & & \\
\hline
x^2+ax & & & & \\
\quad +ax+a^2 & & & & \\
\hline
x^2+2ax+a^2=\textit{ſquare} & & & & \\
x+a & & & & \\
\hline
x^3+2ax^2+a^2x & & & & \\
\quad +ax^2+2a^2x+a^3 & & & & \\
\hline
x^3+3ax^2+3a^2x+a^3=\textit{cube} & & & & \\
x+a & & & & \\
\hline
x^4+3ax^3+3a^2x^2+a^3x & & & & \\
\quad +ax^3+3a^2x^2+3a^3x+a^4 & & & & \\
\hline
x^4+4ax^3+6a^2x^2+4a^3x+a^4=\textit{4th power.} & & & &
\end{array}$$

EXAMPLES FOR PRACTICE.

1. Required the cube or third power of $2a^2$.
   Anſ. $8a^6$.
2. Required the 4th power of $2a^2x$. *Anſ.* $16a^8x^4$.
3. To find the 3d power of $-8x^2y^3$.
   Anſ. $512x^6y^9$.

4. To find the biquadrate of $-\frac{2a^2x}{3b^2}$.

Anſ. $\frac{16a^8x^4}{81b^8}$.

5. Required the 5th power of $a-x$.

Anſ. $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$.

SIR ISAAC NEWTON's RULE *for raiſing a binomial or reſidual quantity to any power whatever.*

1. *To find the terms without the co-efficients.* The index of the firſt, or leading quantity, begins with that of the given power, and decreaſes continually by 1, in every term to the laſt; and in the following quantity the indices of the terms are 0, 1, 2, 3, 4, &c.

2. *To find the unciæ or co-efficients.* The firſt is always 1, and the ſecond is the index of the power: and in general, if the co-efficient of any term be multiplied by the index of the leading quantity, and the product be divided by the number of terms to that place, it will give the co-efficient of the term next following.

Note, The whole number of terms will be one more than the index of the given power; and when both terms of the root are +, all the terms of the power will be +; but if the ſecond term be —, then all the odd terms will be +, and the even terms —.

EXAMPLES:

1. Let $a+x$ be involved to the fifth power.

The terms without the co-efficients will be

a^5, a^4x, a^3x^2, a^2x^3, ax^4, x^5, *and the co-efficients will be*

$$1,\ 5,\ \frac{5\times4}{2},\ \frac{10\times3}{3},\ \frac{10\times2}{4},\ \frac{5\times1}{5};$$

or 1, 5, 10, 10, 5, 1,

And therefore the 5*th power is*

$$a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5.$$

2. Let $x-a$ be involved to the 6th power.

The terms without the co-efficients will be

$$x^6,\ x^5a,\ x^4a^2,\ x^3a^3,\ x^2a^4,\ xa^5,\ a^6,$$

and the co-efficients will be

$$1,\ 6,\ \frac{6\times5}{2},\ \frac{15\times4}{3},\ \frac{20\times3}{4},\ \frac{15\times2}{5},\ \frac{6\times1}{6},$$

or 1, 6, 15, 20, 15, 6, 1,

And therefore the ſixth power of $x-a$ *is*

$$x^6-6x^5a+15x^4a^2-20x^3a^3+15x^2a^4-6xa^5+a^6.$$

3. Find the 4th power of $x-a$.

Anſ. $x^4-4x^3a+6x^2a^2-4xa^3+a^4$.

4. Find the 7th power of $x+b$.

Anſ. $x^7+7x^6a+21x^5a^2+35x^4a^3+35x^3a^4+21x^2a^5+7xa^6+a^7$.

EVOLUTION.

Evolution is the reverſe of involution, and teaches to find the roots of any given powers.

CASE I.

To find the roots of ſimple quantities.

RULE.

Extract the root of the co-efficient for the numerical part, and divide the index of the letters by the index of the power, and it will give the root required.

EXAMPLES:

1. The ſquare root of $9x^2=3x^{\frac{2}{2}}=3x$.
2. The cube root of $8x^3=2x^{\frac{3}{3}}=2x$.
3. The ſquare root of $3a^2x^6=a^{\frac{2}{2}}x^{\frac{6}{2}}\sqrt{3}=ax^3\sqrt{3}$.
4. The cube root of $-125a^3x^6=5a^{\frac{3}{3}}x^{\frac{6}{3}}=5ax^2$.
5. The biquadrate root of $16a^4x^8=2a^{\frac{4}{4}}x^{\frac{8}{4}}=2ax^2$.

CASE II.

To find the ſquare root of a compound quantity.

RULE.

1. Range the quantities according to the dimenſions of ſome letter, and ſet the root of the firſt term in the quotient.

2. Subtract the ſquare of the root, thus found, from the firſt term, and bring down the two next terms to the remainder for a dividend.

3. Divide the dividend by double the root, and ſet the reſult in the quotient.

4. Multiply the diviſor and quotient by the term laſt put in the quotient, and ſubtract the product from the dividend, and ſo on, as in common arithmetic.

EXAMPLES:

1. Extract the ſquare root of $4a^4+12a^3x+13a^2x^2+6ax^3+x^4$.

$$\begin{array}{rl}
 & 4a^4+12a^3x+13a^2x^2+6ax^3+x^4(2a^2+3ax+x^2 \\
 & 4a^4 \\ \hline
4a^2+3ax) & 12a^3x+13a^2x^2 \\
 & 12a^3x+\ \ 9a^2x^2 \\ \hline
4a^2+6ax+x^2) & 4a^2x^2+6ax^3+x^4 \\
 & 4a^2x^2+6ax^3+x^4 \\ \hline
\end{array}$$

*

2. Extract the ſquare root of $x^4-4x^3+6x^2-4x+1$.

$$\begin{array}{rl}
 & x^4-4x^3+6x^2-4x+1(x^2-2x+1 \\
 & x^4 \\ \hline
2x^2-2x) & -4x^3+6x^2 \\
 & -4x^3+4x^2 \\ \hline
2x^2-4x+1) & 2x^2-4x+1 \\
 & 2x^2-4x+1 \\ \hline
\end{array}$$

*

3. Required the ſquare root of $a^4+4a^3x+6a^2x^2+4ax^3+x^4$. *Anſ.* $a^2+2ax+x^2$.

4. Required the ſquare root of $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$. *Anſ.* $x^2-x+\frac{1}{4}$.

5. Required the ſquare root of a^2+x^2.

Anſ. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-$ &c.

CASE III.

To find the roots of powers in general.

RULE.

1. Find the root of the firſt term, and place it in the quotient.

2. Subtract the power, and bring down the ſecond term for a dividend.

3. Involve the root, laſt found, to the next loweſt power, and multiply it by the index of the given power for a diviſor.

4. Divide the dividend by the diviſor, and the quotient will be the next term of the root.

5. Involve the whole root, and ſubtract and divide as before; and ſo on till the whole is finiſhed.

EXAMPLES:

1. Required the ſquare root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$.

$$\begin{array}{l} \qquad a^4-2a^3x+3a^2x^2-2ax^3+x^4(a^2-ax+x^2 \\ \qquad a^4 \\ \hline 2a^2)-2a^3x \\ \hline \qquad a^4-2a^3x+\ a^2x^2 \\ \hline \qquad\qquad 2a^2)2a^2x^2 \\ \hline \qquad a^4-2a^3x+3a^2x^2-2ax^3+x^4 \\ \hline \qquad\qquad\qquad * \end{array}$$

2. Extract the cube root of $x^6+6x^5-40x^3+96x-64$.

$x^6+6x^5-40x^3+96x-64(x^2+2x-4$

$\underline{x^6}$

$3x^4)\ 6x^5$

$\underline{x^6+6x^5+12x^4+8x^3}$

$3x^4)-12x^4$

$\underline{x^6+6x^5-40x^3+96x-64}$

*

3. Required the ſquare root of $a^2+2ab+2ac+b^2+2bc+c^2$. *Anſ.* $a+b+c$.

4. Required the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$. *Anſ.* x^2-2x+1.

5. Required the biquadrate root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$. *Anſ.* $2a-3x$.

SURDS.

Surds are ſuch quantities as have no exact root, and are uſually expreſſed by fractional indices: thus, the ſquare root of 2, and the cube root of 3, &c. cannot be exactly determined, but may be denoted by $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$ &c.

PROBLEM I.

To reduce a rational quantity to the form of a Surd.

RULE.

Multiply the index of the rational quantity by the index of the ſurd, and over this new quantity place the radical ſign, and it will be of the form required.

EXAMPLES:

1. 3 reduced to the form of $\sqrt{7}$ is $\sqrt{3^{1\times2}}=\sqrt{3^2}=\sqrt{9}$.

2. x reduced to the form of $\sqrt[3]{a}$ is $\sqrt[3]{x^{1\times3}}=\sqrt[3]{x^3}$.

3. $a+b$ reduced to the form of $\sqrt{cx}$ is $\sqrt{\overline{a+b}|^2}=\sqrt{a^2+2ab+b^2}$.

4. $\frac{a}{b\sqrt{x}}$ reduced to the form of $\sqrt{c}$ is $\sqrt{\frac{a^2}{b^2x}}$.

PROBLEM II.

To reduce quantities of different indices to other equivalent ones, that ſhall have a common index.

RULE.

1. Divide the indices of the quantities by the given index, and the quotients will be the new indices for thoſe quantities.

2. Over the ſaid quantities, with their new indices, place the given index, and they will make the equivalent quantities required.

EXAMPLES:

1. Reduce $15^{\frac{1}{4}}$ and $9^{\frac{1}{6}}$ to equivalent quantities having the common index $\frac{1}{8}$.

$\frac{1}{4}\div\frac{1}{2}=\frac{1}{4}\times\frac{2}{1}=\frac{2}{4}=\frac{1}{2}=$ *1ſt index.*
$\frac{1}{6}\div\frac{1}{2}=\frac{1}{6}\times\frac{2}{1}=\frac{2}{6}=\frac{1}{3}=$ *2d index*

Therefore $\overline{15^{\frac{1}{2}}}\big|^{\frac{1}{2}}$ and $\overline{9^{\frac{1}{3}}}\big|^{\frac{1}{2}}=$ *quantities required.*

2. Reduce a^2 and $x^{\frac{1}{4}}$ to the ſame common index $\frac{1}{3}$.

$\frac{2}{1}\div\frac{1}{3}=\frac{2}{1}\times\frac{3}{1}=\frac{6}{1}=$ *1ſt index*
$\frac{1}{4}\div\frac{1}{3}=\frac{1}{4}\times\frac{3}{1}=\frac{3}{4}=$ *2d index*

Therefore $\overline{a^6}\big|^{\frac{1}{3}}$ and $\overline{x^{\frac{3}{4}}}\big|^{\frac{1}{3}}=$ *quantities required.*

3. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.

Anſ. $\overline{27}\big|^{\frac{1}{6}}$ *and* $\overline{4}\big|^{\frac{1}{6}}$

4. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{4}}$ to the common index $\frac{1}{8}$.

Anſ. $\overline{a^{\frac{6}{3}}}\big|^{\frac{1}{8}}$ *and* $\overline{b}\big|^{\frac{1}{8}}$.

5. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to the ſame radical ſign.

Anſ. ${}^{nm}\sqrt{a^m}$ *and* ${}^{nm}\sqrt{b^n}$.

PROBLEM III.

To reduce ſurds to their moſt ſimple terms.

RULE.

Find the root of the greateſt power contained in the given ſurd, and ſet it before the remaining quantities, with the proper radical ſign between them.

EXAMPLES:

1. $\sqrt{48}=\sqrt{\overline{3\times16}}=\sqrt{16}\times\sqrt{3}=4\sqrt{3}$.
2. $\sqrt{75}=\sqrt{25\times3}=\sqrt{25}\times\sqrt{3}=5\sqrt{3}$.

3. $\sqrt[3]{81}=\sqrt[3]{\overline{27\times3}}=\sqrt[3]{27}\times\sqrt[3]{3}=3\sqrt[3]{3}$.

4. $\sqrt{49a^2x}=\sqrt{\overline{49a^2\times x}}=\sqrt{49a^2}\times\sqrt{x}=7a\sqrt{x}$.

5. $\left.\overline{bx+x^2}\right|^{\frac{1}{2}}=x^{\frac{1}{2}}\times\left.\overline{b+x}\right|^{\frac{1}{2}}$; $\left.\overline{a^3x-a^2x^2}\right|^{\frac{1}{2}}=a\times\left.\overline{ax-x^2}\right|^{\frac{1}{2}}$.

6. $\left.\overline{\frac{64a^4x^3}{27a-27x}}\right|^{\frac{1}{3}}=\frac{4ax}{3}\times\left.\overline{\frac{a}{a-x}}\right|^{\frac{1}{3}}$.

PROBLEM IV.

To add ſurd quantities together.

RULE.

1. Reduce thoſe quantities that have unlike indices to equivalent ones, having a common index.

2. Bring all fractions to a common denominator, and reduce the quantities to their ſimpleſt terms, as in the laſt problem.

3. Then, if the ſurd part be the ſame in them all, annex it to the ſum of the rational parts, with the ſign of multiplication, and it will give the total ſum required.

If the ſurd part be not the ſame in all the quantities, they can only be added by the ſigns + and —.

EXAMPLES:

1. $2\sqrt{5}+7\sqrt{5}=\overline{2+7}\times\sqrt{5}=9\sqrt{5}$.

2. $\sqrt{27}+\sqrt{48}=3\sqrt{3}+4\sqrt{3}=7\sqrt{3}$.

3. $\overline{500}\rvert^{\frac{1}{3}}+\overline{108}\rvert^{\frac{1}{3}}=\overline{125\times4}\rvert^{\frac{1}{3}}+\overline{27\times4}\rvert^{\frac{1}{3}}=5\times4^{\frac{1}{3}}$
$+3\times4^{\frac{1}{3}}=8\times4^{\frac{1}{3}}$.

4. $\sqrt{27a^4x}+\sqrt{3a^2x^3}=\sqrt{9a^4\times3x}+\sqrt{a^2x^2\times3x}$
$=3a^2\sqrt{3x}+ax\sqrt{3x}=\overline{3a+x}\times a\times\sqrt{3x}$.

5. $\sqrt{\frac{24}{25}}+\sqrt{\frac{2}{3}}=\sqrt{\frac{72}{75}}+\sqrt{\frac{50}{75}}=\sqrt{\frac{36\times2}{75}}+$
$\sqrt{\frac{25\times2}{75}}=6\sqrt{\frac{2}{75}}+5\sqrt{\frac{2}{75}}=11\sqrt{\frac{2}{75}}$.

6. $\sqrt[3]{\frac{1}{32}}+\sqrt[3]{\frac{64}{108}}=\sqrt[3]{\frac{1}{8\times4}}+\sqrt[3]{\frac{64\times1}{27\times4}}=\frac{1}{2}$
$\times\sqrt[3]{\frac{1}{4}}+\frac{4}{3}\times\sqrt[3]{\frac{1}{4}}=\overline{\frac{1}{2}+\frac{4}{3}}\times\sqrt[3]{\frac{1}{4}}=\frac{11}{6}\times$
$\sqrt[3]{\frac{1}{4}}$.

7. $\sqrt{\frac{a}{b}}+\sqrt{\frac{b^3}{a^3}}=\sqrt{\frac{a^4}{ba^3}}+\sqrt{\frac{b^4}{ba^3}}=\frac{a^2}{a}\times\sqrt{\frac{1}{ba}}+$
$\frac{b^2}{a}\times\sqrt{\frac{1}{ba}}=\frac{a^2+b^2}{a}\times\sqrt{\frac{1}{ba}}=\frac{a^2+b^2}{a\sqrt{ba}}$.

PROBLEM V.

To ſubtract ſurd quantities.

RULE.

Prepare the quantities as in the laſt rule, and annex the difference of the rational parts to the common ſurd, with the ſign of multiplication.

EXAMPLES:

1. $3\sqrt{18}-\sqrt{18}=\overline{3-1}\times\sqrt{18}=2\sqrt{18}$.

2. $2\sqrt{50}-\sqrt{18}=2\sqrt{25\times2}-\sqrt{9\times2}=2\times5\times\sqrt{2}-1\times3\times\sqrt{2}=7\sqrt{2}$.

3. $\sqrt[3]{192}-\sqrt[3]{24}=\sqrt[3]{64\times3}-\sqrt[3]{8\times3}=4\times3^{\frac{1}{3}}-2\times3^{\frac{1}{3}}=\overline{4-2}\times3^{\frac{1}{3}}=2\times3^{\frac{1}{3}}$.

4. $\sqrt{80a^4x}-\sqrt{20a^2x^3}=\sqrt{10a^4\times5x}-\sqrt{4a^2x^2\times5x}=4a^2\sqrt{5x}-2ax\sqrt{5x}=\overline{4a^2-2ax}\times\sqrt{5x}$.

5. $\sqrt{\frac{24}{25}}-\sqrt{\frac{2}{3}}=\sqrt{\frac{72}{75}}-\sqrt{\frac{50}{75}}=\sqrt{\frac{36\times2}{75}}-\sqrt{\frac{25\times2}{75}}=6\times\sqrt{\frac{2}{75}}-5\times\sqrt{\frac{2}{75}}=\overline{6-5}\times\sqrt{\frac{2}{75}}=\sqrt{\frac{2}{75}}$.

6. $\sqrt[3]{\frac{16}{27}}-\sqrt[3]{\frac{1}{4}}=\frac{4}{3}\times\sqrt[3]{\frac{1}{4}}-\sqrt[3]{\frac{1}{4}}=\overline{\frac{4}{3}-\frac{3}{3}}\times\sqrt[3]{\frac{1}{4}}=\frac{1}{3}\times\sqrt[3]{\frac{1}{4}}$.

7. $\sqrt{\frac{b}{c}}-\sqrt{\frac{c^3}{b^3}}=\sqrt{\frac{b^4}{cb^3}}-\sqrt{\frac{c^4}{cb^3}}=\frac{b^2}{b}\times\sqrt{\frac{1}{cb}}-\frac{c^2}{b}\times\sqrt{\frac{1}{cb}}=\frac{b^2-c^2}{b}\times\sqrt{\frac{1}{cb}}=\frac{b^2-c^2}{b\times\overline{cb}|^{\frac{1}{2}}}$.

PROBLEM VI.

To multiply ſurd quantities together.

RULE.

Reduce the ſurds to the ſame index; and then the product of the rational quantities being annexed to the product of the ſurds, will give the whole product required.

EXAMPLES:

1. $\sqrt{10}\times\sqrt{5}=\sqrt{50}=\sqrt{25\times2}=5\sqrt{2}$.

2. $3\sqrt{8}\times2\sqrt{6}=3\times2\sqrt{8\times6}=6\sqrt{48}=6\sqrt{16\times3}$
$=4\times6\times\sqrt{3}=24\sqrt{3}$.

3. $\frac{2}{3}\sqrt{\frac{1}{8}}\times\frac{3}{4}\sqrt{\frac{7}{10}}=\frac{2}{3}\sqrt{\frac{8}{64}}\times\frac{3}{4}\sqrt{\frac{7}{10}}=\frac{6}{12}$
$\sqrt{\frac{56}{640}}=\frac{1}{2}\sqrt{\frac{4\times14}{64\times10}}=\frac{1}{2}\times\frac{2}{8}\times\sqrt{\frac{14}{10}}=\frac{2}{16}\sqrt{1}$
$\frac{4}{10}=\frac{1}{8}\sqrt{1\frac{2}{5}}$.

4. $\frac{1}{3}\times\sqrt[3]{\frac{3}{2}}\times\frac{1}{5}\times\sqrt[3]{\frac{1}{4}}=\frac{1}{3}\times\sqrt[3]{\frac{3}{2}}\times$
$\frac{1}{5}\times\sqrt[3]{\frac{3}{12}}=\frac{1}{3}\times\frac{1}{5}\times\sqrt[3]{\frac{3\times3}{2\times12}}=\frac{1}{15}\times\sqrt[3]{\frac{9}{24}}$
$=\frac{1}{15}\times\sqrt[3]{\frac{3}{8}}=\frac{1}{15}\times\frac{1}{2}\times3^{\frac{1}{3}}=\frac{1}{30}\times3^{\frac{1}{3}}$.

5. $a^{\frac{1}{2}}\times a^{\frac{1}{3}}=a^{\frac{3}{6}}\times a^{\frac{2}{6}}=a^{\frac{5}{6}}=\overline{a^{5}}|^{\frac{1}{6}}$.

6. $5x^{\frac{1}{2}}y^{\frac{1}{3}}\times6x^{\frac{2}{3}}y^{\frac{3}{4}}=5x^{\frac{6}{12}}y^{\frac{4}{12}}\times6x^{\frac{8}{12}}y^{\frac{9}{12}}=30x^{\frac{14}{12}}$
$y^{\frac{13}{12}}=30\times\overline{x^{14}y^{13}}|^{\frac{1}{12}}$.

7. $\overline{x+y}|^{\frac{1}{2}}\times\overline{x+y}|^{\frac{1}{3}}=\overline{x+y}|^{\frac{3}{6}}\times\overline{x+y}|^{\frac{2}{6}}$
$=\overline{\overline{x+y}^{3}}|^{\frac{1}{6}}\times\overline{\overline{x+y}^{2}}|^{\frac{1}{6}}=\overline{\overline{x+y}^{5}}|^{\frac{1}{6}}$.

8. $\frac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}}\times\frac{x^{\frac{1}{3}}}{y^{\frac{1}{2}}}=\frac{x^{\frac{3}{6}}}{y^{\frac{2}{6}}}\times\frac{x^{\frac{2}{6}}}{y^{\frac{3}{6}}}=\overline{\frac{x^3}{y^2}}\Big|^{\frac{1}{6}}\times\overline{\frac{x^2}{y^3}}\Big|^{\frac{1}{6}}=\overline{\frac{x^5}{y^5}}\Big|^{\frac{1}{6}}$.

9. $x^{\frac{1}{n}}\times y^{\frac{1}{m}}=x^{\frac{m}{nm}}\times y^{\frac{n}{nm}}=\overline{x^{n}}\big|^{\frac{1}{nm}}\times\overline{y^{n}}\big|^{\frac{1}{nm}}=\overline{x^{m}y^{n}}\big|^{\frac{1}{nm}}$.

10. $\overline{x+\sqrt{y}}\times\overline{x-\sqrt{y}}=x^2-y$.

11. $\overline{x-x\sqrt{a}}\times\overline{y-y\sqrt{a}}=xy-2xy\sqrt{a}+axy$.

12. $\sqrt{a+\sqrt{b-\sqrt{3}}}\times\sqrt{a-\sqrt{b-\sqrt{3}}}$
$=\sqrt{a^2-b+\sqrt{3}}$.

PROBLEM VII.

To divide one ſurd quantity by another.

RULE.

Reduce the ſurds to the ſame index; and then the quotient of the rational quantities being annexed to the quotient of the ſurds, will give the whole quotient required.

EXAMPLES:

1. $\sqrt{375}\div\sqrt{5}=\sqrt{\frac{375}{5}}=\sqrt{75}=\sqrt{\overline{25\times3}}=5\sqrt{3}$.

2. $8\sqrt{108}\div2\sqrt{6}=\frac{8}{2}\sqrt{\frac{108}{6}}=4\sqrt{18}=4\sqrt{\overline{9\times2}}$
$=12\sqrt{2}$.

3. $\frac{3}{4}\sqrt{\frac{1}{135}}\div\frac{2}{3}\sqrt{\frac{1}{5}}=\frac{3}{4}\times\frac{3}{2}\times\sqrt{\overline{\frac{1}{135}\times\frac{5}{1}}}$ $=\frac{9}{8}\sqrt{\frac{5}{135}}=\frac{9}{8}\sqrt{\frac{1}{9\times3}}=\frac{3}{8}\sqrt{\frac{1}{3}}.$

4. $\frac{5}{7}\times\sqrt[3]{\frac{8}{189}}\div\frac{2}{5}\times\sqrt[3]{\frac{8}{7}}=\frac{5}{7}\times\frac{5}{2}\times$ $\left.\overline{\frac{8}{189}\times\frac{7}{8}}\right|^{\frac{1}{3}}=\frac{25}{42}\times\left.\overline{\frac{1}{3}}\right|^{\frac{1}{3}}.$

5. $x^{\frac{1}{2}}\div y^{\frac{1}{3}}=x^{\frac{3}{6}}\div y^{\frac{2}{6}}=\left.\overline{x^3}\right|^{\frac{1}{6}}\div\left.\overline{y^2}\right|^{\frac{1}{6}}=\left.\overline{\frac{x^3}{y^2}}\right|^{\frac{1}{6}}.$

6. $\frac{2}{5}x^{\frac{1}{2}}y^{\frac{1}{2}}\div\frac{3}{4}x^{\frac{1}{3}}y^{\frac{1}{3}}=\frac{2}{5}x^{\frac{3}{6}}y^{\frac{3}{6}}\div\frac{3}{4}x^{\frac{2}{6}}y^{\frac{2}{6}}=\frac{2}{5}\times$ $\frac{4}{3}\times x^{\frac{1}{6}}y^{\frac{1}{6}}=\frac{8}{15}\times\left.\overline{xy}\right|^{\frac{1}{6}}.$

7. $\left.\overline{x+y}\right|^{\frac{1}{2}}\div\left.\overline{x+y}\right|^{\frac{1}{3}}=\left.\overline{x+y}\right|^{\frac{3}{6}}\div\left.\overline{x+y}\right|^{\frac{2}{6}}=\left.\overline{x+y}\right|^{\frac{1}{6}}.$

8. $\frac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}}\div\frac{y^{\frac{1}{2}}}{x^{\frac{1}{3}}}=\frac{x^{\frac{3}{6}}}{y^{\frac{2}{6}}}\times\frac{x^{\frac{2}{6}}}{y^{\frac{3}{6}}}=\left.\overline{\frac{x^5}{y^5}}\right|^{\frac{1}{6}}.$

9. $x^{\frac{1}{n}}\div x^{\frac{1}{m}}=x^{\frac{m}{nm}}\div x^{\frac{n}{nm}}=x^{\frac{m-n}{nm}}=\left.\overline{x^{m-n}}\right|^{\frac{1}{nm}}$

10. $\overline{x^2-xd-b+d\sqrt{b}}\div\overline{x-\sqrt{b}}=x+\sqrt{b}-d.$

PROBLEM VIII.

To involve ſurd quantities to any power.

RULE.

Multiply the index of the quantity by the index of the power to be raiſed, and to the reſult annex the power of the rational parts, and it will be involved as required.

EXAMPLES:

1. The cube of $\sqrt{2}=2^{\frac{1}{2}\times3}=2^{\frac{3}{2}}=\left.\overline{2^3}\right|^{\frac{1}{2}}=8^{\frac{1}{2}}$.

2. The ſquare of $\frac{2}{3}a^{\frac{1}{3}}=\frac{4}{9}a^{\frac{1}{3}\times2}=\frac{4}{9}a^{\frac{2}{3}}=\frac{4}{9}\left.\overline{a^2}\right|^{\frac{1}{3}}$.

3. The 4th power of $\frac{1}{2}\sqrt{\frac{2}{3}}=\frac{1}{16}\times\left.\overline{\frac{2}{3}}\right|^{\frac{1}{2}\times4}=\frac{1}{16}\times\left.\overline{\frac{2}{3}}\right|^{\frac{4}{2}}=\frac{1}{16}\times\left.\overline{\frac{2}{3}}\right|^{2}=\frac{1}{36}$.

4. The ſquare of $\frac{a}{b}\times\left.\overline{\frac{x}{c}}\right|^{\frac{1}{3}}=\frac{a^2}{b^2}\times\left.\overline{\frac{x}{c}}\right|^{\frac{1}{3}\times2}=\frac{a^2}{b^2}\times\left.\overline{\frac{x}{c}}\right|^{\frac{2}{3}}=\frac{a^2}{b^2}\times\left.\overline{\frac{x^2}{c^2}}\right|^{\frac{1}{3}}$.

5. The *n*th power of $a^{\frac{1}{m}}=a^{\frac{1}{m}\times n}=a^{\frac{n}{m}}=\left.\overline{a^n}\right|^{\frac{1}{m}}$.

6. The *n*th power of $\left.\overline{x+y}\right|^{\frac{1}{n}}=\left.\overline{x+y}\right|^{\frac{n}{m}}=\left.\overline{\overline{x+y}^{\,n}}\right|^{\frac{1}{n}}$.

7. The ſquare of $3+\sqrt{5}=14+6\sqrt{5}$.

8. The cube of $x-\sqrt{y}=x^3-3x^2\sqrt{y}+3xy-y\sqrt{y}$.

PROBLEM IX.

To find the roots of ſurd quantities.

RULE.

Divide the index of the given quantity by the index of the root to be extracted, and to the reſult add the root of the rational part, and it will give the root required.

EXAMPLES:

1. The ſquare root of $3^{\frac{1}{3}} = 3^{\frac{1}{3} \div 2} = 3^{\frac{1}{6}}$.
2. The cube root of $\frac{8}{27} \times a^3 = \frac{2}{3} \times a^{3 \div 3} = \frac{2}{3} a$.
3. The cube root of $\frac{a}{b} \times \left.\overline{\frac{x}{y}}\right|^{\frac{1}{2}} = \left.\overline{\frac{a}{b}}\right|^{\frac{1}{3}} \times \left.\overline{\frac{x}{y}}\right|^{\frac{1}{6}}$
$= \left.\overline{\frac{a}{b}}\right|^{\frac{2}{6}} \times \left.\overline{\frac{x}{y}}\right|^{\frac{1}{6}} = \left.\overline{\frac{a^2x}{b^2y}}\right|^{\frac{1}{6}}$.
4. The 4th root of $3x^2 = x^{\frac{2}{4}} \times 3^{\frac{1}{4}} = 3^{\frac{1}{4}} \times x^{\frac{1}{2}}$.
5. The nth root of $x^{\frac{r}{m}} = x^{\frac{r}{nm}} = \left.\overline{x^r}\right|^{\frac{1}{nm}}$
6. The nth root of $\overline{x+y}^{\frac{n}{m}} = \left.\overline{x+y}\right|^{\frac{n}{nm}} = \left.\overline{x+y}\right|^{\frac{1}{n}}$
7. The ſquare root of $x^2 - 4x\sqrt{a} + 4a = x - 2\sqrt{a}$.
8. The nth root of $x - 2\sqrt{\overline{x+y}} = \left.\overline{x - 2\sqrt{x+y}}\right|^{\frac{1}{n}}$.

Of INFINITE SERIES.

An infinite ſeries is formed from a vulgar fraction, having a compound denominator, or by extracting the root of a ſurd quantity; and is ſuch, as, being continued, would run on *ad infinitum*, in the manner of a decimal fraction. And, by obtaining a few of the firſt terms, the law of the progreſſion will be manifeſt, ſo that the ſeries may be continued, without actually performing the whole operation.

PROBLEM I.

To reduce fractional quantities into infinite ſeries.

RULE.

Divide the numerator by the denominator, as in common diviſion; and the operation continued, as far as ſhall be thought neceſſary, will give the ſeries required.

EXAMPLES:

1. Let $\frac{ax}{a-x}$ be propoſed to be thrown into an infinite ſeries.

$$
\begin{array}{r@{}l}
a-x)\,& ax\ldots\left(x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}\ \&c.\right.\\
& \underline{ax-x^2}\\
& \qquad x^2\\
& \qquad \underline{x^2-\frac{x^3}{a}}\\
& \qquad\qquad \frac{x^3}{a}\\
& \qquad\qquad \underline{\frac{x^3}{a}-\frac{x^4}{a^2}}\\
& \qquad\qquad\qquad \frac{x^4}{a^2}\\
& \qquad\qquad\qquad \underline{\frac{x^4}{a^2}-\frac{x^5}{a^3}}\\
& \qquad\qquad\qquad\qquad \frac{x^5}{a^3}\ \&c.
\end{array}
$$

2. Let $\frac{a^2}{a^2+2ax+x^2}$ be propoſed to be thrown into an infinite ſeries.

$$a^2+2ax+x^2)\,a^2 \quad . \quad . \quad . \quad (1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}\ \&c.$$

$$\underline{a^2+2ax+x^2}$$

$$-2ax-x^2$$

$$\underline{-2ax-4x^2-\frac{2x^3}{a}}$$

$$3x^2+\frac{2x^3}{a}$$

$$\underline{3x^2+\frac{6x^3}{a}+\frac{3x^4}{a^2}}$$

$$-\frac{4x^3}{a}-\frac{3x^4}{a^2}$$

$$\underline{-\frac{4x^3}{a}-\frac{8x^4}{a^2}-\frac{4x^5}{a^3}}$$

$$\frac{5x^4}{a^2}+\frac{4x^5}{a^3}\ \&c.$$

3. To throw $\frac{b}{a+x}$ into an infinite ſeries.

Anſ. $\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}+$ &c.

4. To throw $\frac{a^2}{x+b}$ into an infinite ſeries.

Anſ. $\frac{a^2}{x}-\frac{a^2b}{x^2}+\frac{a^2b^2}{x^3}-\frac{a^2b^3}{x^4}+$ &c.

5. To throw $\frac{1}{1+x^2}$ into an infinite ſeries.

Anſ. $1-x^2+x^4-x^6+x^8-$ &c.

6. To throw $\dfrac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}$ into an infinite ſeries.

Anſ. $2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^{2}+34x^{\frac{5}{2}}$ &c.

PROBLEM II.

To reduce a compound ſurd into an infinite ſeries.

RULE.

Extract the root as in common arithmetic, and the operation, continued as far as ſhall be thought neceſſary, will give the ſeries required.

EXAMPLES:

1. Extract the ſquare root of a^2+x^2 in an infinite ſeries.

$$a^2+x^2\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}\ \text{\&c.}\right.$$

$$\underline{a^2}$$

$$2a+\frac{x^2}{2a}\Big)\,x^2$$

$$\underline{x^2+\frac{x^4}{4a^2}}$$

$$2a+\frac{x^2}{a}-\frac{x^4}{8a^3}\Big)-\frac{x^4}{4a^2}$$

$$\underline{-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}}$$

$$2a+\frac{x^2}{a}-\frac{x^4}{8a^3}\ \text{\&c.}\Big)\,\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$$

$$\underline{\frac{x^6}{8a^4}+\frac{x^8}{16a^6}\ \text{\&c.}}$$

$$-\frac{5x^8}{64a^6}\ \text{\&c.}$$

2. Throw $\overline{x-x^2}|^{\frac{1}{2}}$ into an infinite ſeries.

Anſ. $x^{\frac{1}{2}}-\frac{x^{\frac{3}{2}}}{2}-\frac{x^{\frac{5}{2}}}{8}-\frac{x^{\frac{7}{2}}}{16}-\frac{5x^{\frac{9}{2}}}{128}$ &c.

3. Throw $\overline{a^2-x^2}|^{\frac{1}{2}}$ into an infinite ſeries.

Anſ. $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^5}{16a^5}$ &c.

4. Throw $\overline{1-x^3}|^{\frac{1}{3}}$ into an infinite ſeries.

Anſ. $1-\frac{x^3}{3}-\frac{x^6}{9}-\frac{5x^9}{81}$ &c.

PROBLEM III.

To reduce a binomial ſurd into an infinite ſeries; or to extract any root of a binomial.

RULE.

Subſtitute the particular letters of the binomial, with their proper ſigns, in the following general form, and it will give the root required; obſerving that P is the firſt term, Q the ſecond term divided by the firſt, $\frac{m}{n}$ the index of the power or root; and A, B, C, D, &c. the foregoing terms, with their ſigns.

$$\overline{P+PQ}|^{\frac{m}{n}}=P^{\frac{m}{n}}\ (A)+\frac{m}{n}AQ\,(B)+\frac{m-n}{2n}BQ\,(C)+\frac{m-2n}{3n}CQ\,(D)+\frac{m-3n}{4n}DQ\,(E)\ \&c.$$

EXAMPLES:

1. To extract the ſquare root of r^2-x^2, in an infinite ſeries.

Here $P = r^2$, $Q = \frac{-x^2}{r^2}$, and $\frac{m}{n} = \frac{1}{2}$.

Therefore $\overline{r^2-x^2}\big|^{\frac{1}{2}} = r + \frac{1}{2}A \times \frac{-x^2}{r^2} - \frac{1}{4}B \times \frac{-x^2}{r^2} - \frac{3}{6}C \times \frac{-x^2}{r^2} - \frac{5}{8}D \times \frac{-x^2}{r^2}$ &c. $= r - \frac{x^2}{2r^2}A + \frac{x^2}{4r^2}B + \frac{3x^2}{6r^2}C + \frac{5x^2}{8r^2}D$ &c.

And, by reſtoring the values of A, B, C, D, &c. *we ſhall have* $\overline{r^2-x^2}\big|^{\frac{1}{2}} = r - \frac{x^2}{2r} - \frac{x^4}{8r^3} - \frac{x^6}{16r^5} - \frac{5x^8}{128r^7}$ &c. = *ſeries required.*

2. To extract the cube root of $\frac{a^2}{\overline{a^2+x^2}\big|^2}$ in an infinite ſeries.

This expreſſion, reduced, is $\frac{a^{\frac{2}{3}}}{\overline{a^2+x^2}\big|^{\frac{2}{3}}} = a^{\frac{2}{3}} \times \overline{a^2+x^2}\big|^{-\frac{2}{3}}$;

and therefore $P = a^2$, $Q = \frac{x^2}{a^2}$, *and* $\frac{m}{n} = \frac{-2}{3}$.

Whence $\overline{a^2+x^2}\big|^{-\frac{2}{3}} = a^{2^{-\frac{2}{3}}} - \frac{2}{3}A \times \frac{x^2}{a^2} - \frac{5}{6}B \times \frac{x^2}{a^2} - \frac{8}{9}C \times \frac{x^2}{a^2} - \frac{11}{12}D \times \frac{x^2}{a^2}$ &c. $= \frac{1}{\sqrt[3]{a}} - \frac{2x^2}{3\sqrt[3]{a^3}} + \frac{5x^4}{9\sqrt[3]{a^5}} - \frac{40x^6}{81\sqrt[3]{a^7}} + \frac{110x^8}{243\sqrt[3]{a^9}}$ &c. $= \frac{1}{\sqrt[3]{a}} \times : \frac{1}{a} - \frac{2x^2}{3a^3} - \frac{5x^4}{9a^5} - \frac{40x^6}{81a^7} + \frac{110x^8}{243a^9}$ &c.

Therefore $\frac{a^{\frac{2}{3}}}{\overline{a^2+x^2}\big|^{\frac{2}{3}}} = \frac{1}{a^{\frac{1}{3}}} \times : 1 - \frac{2x^2}{3a^2} + \frac{5x^4}{9a^4} - \frac{40x^6}{81a^6}$ &c. *as required.*

3. To find the value of $\frac{r^2}{r+x}$, in an infinite ſeries.

Anſ. $r-x+\frac{x^2}{r}-\frac{x^3}{r^2}+\frac{x^4}{r^3}$ &c.

4. To find the value of $\frac{1}{\overline{a^2-x^2}\big|^{\frac{1}{2}}}$ in an infinite ſeries.

Anſ. $\frac{1}{a}+\frac{x^2}{2a^3}+\frac{3x^4}{8a^5}+\frac{15x^6}{48a^7}$ &c.

5. To find the value of $\frac{a^2}{\overline{a+x}\big|^{2}}$ in an infinite ſeries.

Anſ. $1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+\frac{5x^4}{a^4}$ &c.

6. To find the value of $\frac{1}{\overline{2rx-x^2}\big|^{\frac{1}{2}}}$ in an infinite ſeries.

Anſ. $\frac{1}{\sqrt{2rx}}\times:1+\frac{x}{4r}+\frac{3x^2}{4.8r^2}+\frac{3.5x^3}{4.8.12r^3}$ &c.

7. Find the value of $\overline{a^2-x^2}\big|^{\frac{1}{5}}$, in an infinite ſeries.

Anſ. $a^{\frac{2}{5}}\times:1-\frac{x^2}{5a^2}-\frac{2x^4}{25a^4}-\frac{6x^6}{125a^6}$ &c.

8. To find the value of $\overline{1-x}\big|^{\frac{1}{4}}$ in an infinite ſeries.

Anſ. $1-\frac{x}{4}-\frac{3x^2}{4.8}-\frac{3.7x^3}{4.8.12}-\frac{3.7.11\,x^4}{4.8.12.16}$ &c.

9. To find the value of $\overline{\frac{a^2+x^2}{a^2-x^2}}\Big|^{\frac{1}{2}}$ in a ſeries.

Anſ $1+\frac{x^2}{a^2}-\frac{x^4}{2a^2}+\frac{x^6}{2a^6}$ &c.

10. To find the value of $\frac{ax}{a^2-ax+x^2}$ in a ſeries.

Anſ. $\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^4}{a^4}-\frac{x^5}{a^5}$ &c.

ARITHMETICAL PROPORTION.

Arithmetical proportion is the relation which two quantities, of the ſame kind, bear to each other, with reſpect to their difference.

Four quantities are ſaid to be in *arithmetical proportion*, when the difference between the firſt and ſecond is equal to the difference between the third and fourth.

Thus, 3, 7, 12, 16, *and* a, $a+b$, c, $c+b$, *are arithmetically proportional.*

Arithmetical progreſſion is when a ſeries of quantities either increaſe or decreaſe by the ſame common difference.

Thus, 2, 4, 6, 8, 10, 12, &c. *and* a, $a+b$, $a+2b$, $a+3b$, $a+4b$, $a+5b$, &c. *are ſeries in arithmetical progreſſion, whoſe common differences are* 2, *and* b.

The moſt uſeful part of arithmetical proportion is contained in the following theorems:

I. If four quantities be in arithmetical proportion, the ſum of the two means will be equal to the ſum of the two extremes.

Thus, if 2, 5, 7, 10, *and* a, b, c, d, *are in arithmetical proportion, then will* $2+10=5+7$, *and* $a+d=b+c$.

II. In any arithmetical ſeries, the laſt term is equal to the firſt, more the product of the common difference by the number of terms leſs one.

Thus the 20th term of 2, 4, 6, 8, 10, 12, &c. is $=2+\overline{20-1}\times2=2+19\times2=2+38=40$,

And the nth *term of* a, $a+x$, $a+2x$, $a+3x$, $a+4x$, &c. *is* $=a+\overline{n-1}\times x=a+\overline{n-1}x$.

III. The ſum of any ſeries of quantities in arithmetical progreſſion, is equal to the ſum of the two extremes multiplied by half the number of terms.

Thus the ſum of 1, 2, 3, 4, 5, 6, &c. *continued to the 20th term, is* $=\frac{\overline{1+20}\times20}{2}=\frac{21\times20}{2}=21\times10=20=210$.

And the ſum of n *terms of* a, $a+x$, $a+2x$, $a+3x$, &c. *to* $a+mx$, *is* $=\frac{a+a+m\times n}{2}=a+\frac{mn}{2}$.

GEOMETRICAL PROPORTION.

Geometrical proportion is that relation of two quantities, of the ſame kind, which ariſes from conſidering what part the one is of the other, or how often it is contained in it.

In four proportional quantities, the firſt and third are called the *antecedents,* and the ſecond and fourth the *conſequents.*

Ratio is the quotient which ariſes from dividing the antecedent by the conſequent, or the conſequent by the antecedent.

Four quantities are ſaid to be proportional, when the firſt is the ſame part or multiple of the ſecond, as the third is of the fourth.

Thus, 2, 8, 3, 12, *and* a, ar, b, br, *are geometrical proportionals.*

Direct proportion is when the ſame relation ſubſiſts between the firſt term and the ſecond, as between the third and the fourth.

Thus, 3, 6, 5, 10, *and* x, ax, y, ay, *are in direct proportion.*

Reciprocal, or inverſe proportion, is when one quantity increaſes in the ſame proportion that another diminiſhes.

Thus, 2, 6, 9, 3, *and* a, ar, br, b, *are in inverſe proportion.*

A ſeries of quantities are ſaid to be in geometrical progreſſion, when the firſt has the ſame ratio to the ſecond, as the ſecond to the third, and as the third to the fourth, &c.

Thus, 2, 4, 8, 16, 32, 64, &c. *and* a, ar, ar^2, ar^3, ar^4, ar^5, *&c. are ſeries in geometrical progreſſion.*

The moſt neceſſary part of geometrical proportion is contained in the following theorems.

I. If four quantities be in geometrical proportion, the product of the two means will be equal to that of the two extremes.

Thus, if 2, 4, 6, 12, *and* a, ar, b, br, *be geometrically proportional, then will* $2\times12=4\times6$, *and* $a\times br=b\times ar$.

II. If four quantities be in geometrical proportion, the rectangle of the means divided by either of the extremes will give the other extreme.

Thus, if 3, 9, 5, 15, *and* x, ax, y, ay, *are geometricalproportionals, then will* $\frac{9\times5}{3}=15$, *and* $\frac{ax\times y}{ay}=x$.

III. The ſum of any ſeries of quantities in geometrical progreſſion, is found by multiplying the laſt term by the ratio, and dividing the difference of this product and the firſt term by the ratio leſs one.

Thus, the ſum of 2, 4, 8, 16, 32, 64, 128, 256, 512, *is* $\frac{512\times2-2}{2-1}=1024-2=1022$.

And the ſum of n terms of a, ar, ar^2, ar^3, ar^4, *&c. to* ar^{n-1}, *is* $\frac{ar^{n-1}\times r-a}{r-1}=\frac{ar^n-a}{r-1}$.

IV. If four quantities, a, b, c, d, or 2, 6, 5, 15, are proportional, then will any of the following forms of thoſe quantities be alſo proportional.

1. *directly* $a:b::c:d$ or $2:6::5:15$.
2. *inverſely* $b:a::d:c$ or $6:2::15:5$.
3. *alternately* $a:c::b:d$ *or* $2:5+6:15$.
4. *compoundedly* $a:a+b::c:c+d$ *or* $2:8::5:20$.
5. *dividedly* $a:b-a::c:d-c$ *or* $2:4::5:10$.
6. *mixtly* $b+a:b-a::d+c:d-c$ *or* $8:4::20:10$.
7. *by multiplication* $ra:rb::c:d$ *or* $2\times3(6):6\times3(18)::5:15$.

8. *by diviſion* $\frac{a}{r} : \frac{b}{r} :: c : d$ *or* $\frac{2}{2}(1) : \frac{6}{2}(3) :: 5 : 15$.

SIMPLE EQUATIONS.

An equation is, when two equal quantities, differently expreſſed, are compared together, by means of the ſign $=$ placed between them.

Thus $12-5=7$ is an equation, expreſſing the equality of the quantities $12-5$ and 7.

A ſimple equation is that which contains only one unknown quantity, without including its power.

Thus $x-a+b=c$ is a ſimple equation, containing only the unknown quantity x.

Reduction of equations is the method of finding the value of the unknown quantity; which is ſhewn in the following rules.

RULE I.

Any quantity may be tranſpoſed from one ſide of the equation to the other, by changing its ſign.

Thus, if $x+3=7$, *then will* $x=7-3=4$.

And, if $x-4+6=8$, *then will* $x=8+4-6=6$.

Alſo, if $x-a+b=c-d$, *then will* $x=c-d+a-b$.

And, in like manner, if $4x-8=3x+20$, *then will* $4x-3x=20+8$, *or* $x=28$.

RULE II.

If the unknown term be multiplied by any quantity, it may be taken away by dividing all the other terms of the equation by it.

Thus, if $ax=ab-a$, *then will* $x=b-1$.

And, if $2x+4=16$, *then will* $x+2=8$, *and* $x=8-2=6$.

In like manner, if $ax+2ba=3c^2$, *then will* $x+2b=\frac{3c^2}{a}$, *and* $x=\frac{3c^2}{a}-2b$.

RULE III.

If the unknown term be divided by any quantity, it may be taken away by multiplying all the other terms of the equation by it.

Thus, if $\frac{x}{2}=5+3$, *then will* $x=10+6=19$.

And, if $\frac{x}{a}=b+c-d$, *then will* $x=ab+ac-ad$.

In like manner, if $\frac{2x}{3}-2=6+4$, *then will* $2x-6=18+12$, *and* $2x=18+12+6=36$, *or* $x=\frac{36}{2}=18$.

RULE IV.

The unknown quantity in any equation may be made free from ſurds, by tranſpoſing the reſt of the terms according to the rule, and then involving each ſide to ſuch a power as is denoted by the index of the ſaid ſurd.

Thus, if $\sqrt{x}-2=6$, *then will* $\sqrt{x}=6+2=8$, *and* $x=8^2=64$.

And, if $\sqrt{4x+16}=12$, *then will* $4x+16=144$, *and* $4x=144-16=128$, *or* $x=\frac{128}{4}=32$.

In like manner, if $\sqrt[3]{2x+3}+4=8$, *then will* $\sqrt{2x+3}=8-4=4$, *and* $2x+3=4^3=64$, *and* $x=64-3=61$, *or* $x=\frac{61}{2}=30\frac{1}{2}$.

RULE V.

If that ſide of the equation which contains the unknown quantity be a complete power, it may be reduced by extracting the root of the ſaid power from both ſides of the equation.

Thus, if $x^2+6x+9=25$, *then will* $x+3=\sqrt{25}=5$, *or* $x=5-3=2$.

And, if $3x^2-9=21+3$, *then will* $3x^2=21+3+9=33$, *and* $x^2=\frac{33}{3}=11$, *or* $x=\sqrt{11}$.

In like manner, if $\frac{2x^2}{3}+10=20$, *then will* $2x^2+30=60$, *and* $x^2+15=30$, *or* $x^2=30-15=15$, *or* $x=\sqrt{15}$.

RULE VI.

Any analogy, or proportion, may be converted into an equation, by making the product of the two mean terms equal to that of the two extremes.

Thus, if $3x:16::5:10$, *then will* $3x\times10=16\times5$, *and* $30x=80$, *or* $x=\frac{80}{30}=2\frac{2}{3}$.

And, if $\frac{2x}{3}:a::b:c$, *then will* $\frac{2cx}{3}=ab$, *and* $2cx=3ab$, *or* $x=\frac{3ab}{2c}$.

In like manner, if $12-x:\frac{x}{2}::4:1$, *then will* $12-x=\frac{4x}{2}=2x$, *and* $2x+x=12$, *or* $x=\frac{12}{3}=4$.

RULE VII.

If any quantity be found on both ſides of the equation with the ſame ſign, it may be taken away from them both; and if every term in an equation be multiplied or divided by the ſame quantity, it may be ſtruck out of them all.

Thus, if $4x+a=b+a$, *then will* $4x=b$, *and* $x=\frac{b}{4}$.

And, if $3ax+5ab=8ac$, *then will* $3x+5b=8c$, *and* $x=\frac{8c-5b}{3}$.

In like manner, if $\frac{2x}{3}-\frac{8}{3}=\frac{16}{3}-\frac{8}{3}$, *then will* $2x=16$, *and* $x=8$.

MISCELLANEOUS EXAMPLES.

1. Given $5x-15=2x+6$ to find the value of x.
   Firſt $5x-2x=6+15$
   then $3x=21$
   and $x=\frac{21}{3}=7$
2. Given $40-6x-16=120-14x$ to find x.
   Firſt $14x-6x=120-40+16$
   then $8x=96$
   and, therefore $x=\frac{96}{8}=12$.

3. Let $5ax-3b=2dx+c$ be given, to find x.
Firſt, $5ax-2dx=c+3b$
or $\overline{5a-2d}\times x=c+3b$
and therefore $x=\frac{c+3b}{5a-2d}$.

4. Let $3x^2-10x=8x+x^2$ be given to find x.
Firſt, $3x-10=8+x$
and then $3x-x=8+10$
therefore $2x=18$, *and* $x=\frac{18}{2}=9$.

5. Given $6ax^3-12abx^2=3ax^3+6ax^2$, to find x.
Firſt, dividing the whole by $3ax^2$, *we ſhall have*
$2x-4b=x+2$
and then $2x-x=2+4b$
whence $x=2+4b$.

6. Let $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=10$, be given to find x.
Firſt, $x-\frac{2x}{3}+\frac{2x}{4}=20$
and then $3x-2x+\frac{6x}{4}=60$
and $12x-8x+6x=240$
therefore $10x=240$
and $x=\frac{240}{10}=24$.

7. Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x+19}{2}$ to find x.
Firſt, $x-3+\frac{2x}{3}=40-x-19$
and then $3x-9+2x=120-3x-57$
and therefore $3x+2x+3x=120-57+9$
that is $8x=72$, *or* $x=\frac{72}{8}=9$.

8. Let $\sqrt{\frac{2}{3}x}+5=7$, be given to find x.

Firſt, $\sqrt{\frac{2}{3}x}=7-5=2$

and then $\frac{2}{3}x=2^2=4$

and $2x=12$, *or* $x=\frac{12}{2}=6$.

9. Let $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}$ be given to find x.

Firſt, $x\sqrt{a^2+x^2}+a^2+x^2=2a^2$

and then $x\sqrt{a^2+x^2}=a^2-x^2$

and $x^2\times\overline{a^2+x^2}=\overline{a^2-x^2}\big|^2=a^4-2a^2x^2+x^4$

or $a^2x^2+x^4=a^4-2a^2x^2+x^4$

whence $a^2x^2+2a^2x^2=a^4$, *or* $3a^2x^2=a^4$

and conſequently $x^2=\frac{a^4}{3a^2}$, *and* $x=\sqrt{\frac{a^4}{3a^2}}=a\sqrt{\frac{1}{3}}$

EXAMPLES FOR PRACTICE.

1. Given $x+18=3x-5$ to find x. *Anſ.* $x=11\frac{1}{2}$.
2. Given $3y-a+b=cd$ to find y.

   Anſ. $y=\frac{cd+a-b}{3}$.
3. Given $6-2x+10=20+3x-2$ to find x.

   Anſ. $x=2$.
4. Given $3ax+\frac{a}{2}-3=bx-a$ to find x.

   Anſ. $x=\frac{6-3a}{6a-2b}$.

5. Given $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=d$ to find x.

Anſ. $x=\frac{abcd}{bc+ac+ab}$.

6. Given $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{1}{2}$ to find x.

Anſ. $x=\frac{6}{7}$.

7. Given $\sqrt{12+x}=2+\sqrt{x}$ to find x.

Anſ. $x=4$.

8. Given $\sqrt{a^2+x^2}=\overline{b^4+x^4}\big|^{\frac{1}{4}}$ to find x.

Anſ. $x=\sqrt{\frac{b^4-a^4}{2a^2}}$.

9. Given $x+a=\sqrt{a^2+x\sqrt{b^2+x^2}}$ to find x.

Anſ. $x=\frac{b^2}{4a}-a$.

PROBLEM I.

To exterminate two unknown quantities, or to reduce the two ſimple equations containing them to a ſingle one.

RULE I.

1. Obſerve which of the unknown quantities is the leaſt involved, and find its value in each of the equations, by the methods already explained.

2. Let the two values thus found be made equal to each other, and there will ariſe a new equation with only one unknown quantity in it, whoſe value may be found as before.

EXAMPLES:

1. Given $\left\{\begin{array}{l}2x+3y=23\\5x-2y=10\end{array}\right\}$ to find x and y.

From the firſt equation $x=\frac{23-3y}{2}$,

and from the ſecond $x=\frac{10+2y}{5}$,

and conſequently $\frac{23-3y}{2}=\frac{10+2y}{5}$,

or $115-15y=20+4y$,
or $19y=115-20=95$,

and $y=\frac{95}{19}=5$,

whence $x=\frac{23-15}{2}=4$.

2. Given $\left\{\begin{array}{l}x+y=a\\x-y=b\end{array}\right\}$ to find x and y.

From the firſt equation $x=a-y$,
and from the ſecond $x=b+y$,
Therefore $a-y=b+y$, *or* $2y=a-b$,

and conſequently $y=\frac{a-b}{2}$,

and $x=a-y=a-\frac{a-b}{2}=\frac{a+b}{2}$.

3. Given $\left\{\begin{array}{l}\frac{x}{2}+\frac{y}{3}=7\\\frac{x}{3}+\frac{y}{2}=8\end{array}\right\}$ to find x and y.

From the firſt equation $x=14-\frac{2y}{3}$,

and from the ſecond $x=24-\frac{3y}{2}$, *Therefore* $14-\frac{2y}{3}=24-\frac{3y}{2}$,

and $42-2y=72-\frac{9y}{2}$,

or $84-4y=144-9y$;

whence $5y=144-84=60$,

and $y=\frac{60}{5}=12$,

and $x=14-\frac{2y}{3}=14-\frac{24}{3}=6$.

4. Given $4x+y=34$, and $4y+x=16$, to find x and y. *Anſ.* $x=8$, *and* $y=2$.

5. Given $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$, and $\frac{3x}{4}+\frac{2y}{5}=\frac{61}{120}$, to find x and y. *Anſ.* $x=\frac{1}{2}$ *and* $y=\frac{1}{3}$.

6. Given $x+y=s$, and $x^2-y^2=d$, to find x and y. *Anſ.* $x=\frac{s^2+d}{2s}$, *and* $y=\frac{s^2-d}{2s}$.

RULE II.

1. Conſider which of the unknown quantities you would firſt exterminate, and let its value be found in that equation where it is the leaſt involved.

2. Subſtitute the value, thus found, for its equal in the other equation, and there will ariſe a new equation, with only one unknown quantity, whoſe value may be found as before.

EXAMPLES:

1. Given $\left\{\begin{array}{l} x+2y=17 \\ 3x-y=2 \end{array}\right\}$ to find x and y.

From the firſt equation $x=17-2y$,
And this value ſubſtituted for x *in the ſecond, gives*
$\overline{17-2y}\times 3-y=2$,
or $51-6y-y=2$, *or* $51-7y=2$;
that is $7y=51-2=49$;
whence $y=\frac{49}{7}=7$, *and* $x=17-2y=17-14=2$.

2. Given $\left\{\begin{array}{l} x+y=13 \\ x-y=3 \end{array}\right\}$ to find x and y.

From the firſt equation $x=13-y$,
And this value being ſubſtituted for x *in the 2d,*
gives $13-y-y=3$, *or* $13-2y=3$,
that is $2y=13-3=10$,
or $y=\frac{10}{2}=5$, *and* $x=13-y=13-5=8$.

3. Given $\left\{\begin{array}{l} a:b::x:y \\ x^2+y^2=c \end{array}\right\}$ to find x and y.

The firſt analogy turned into an equation
is $bx=ay$, *or* $x=\frac{ay}{b}$,
And this value of x *ſubſtituted in the 2d,*
gives $\left.\overline{\frac{ay}{b}}\right|^2+y^2=c$, *or* $\frac{a^2y^2}{b^2}+y^2=c$,
or $a^2y^2+b^2y^2=cb^2$, *or* $y^2=\frac{cb^2}{a^2+b^2}$,
and therefore $y=\left.\overline{\frac{cb^2}{a^2+b}}\right|^{\frac{1}{2}}$, *and* $x=\left.\overline{\frac{ca^2}{a^2+b^2}}\right|^{\frac{1}{2}}$.

4. Given $\frac{x}{7}+7y=99$, *and* $\frac{y}{7}+7x=51$, to find nd y. *Anſ.* $x=7$ *and* $y=14$.

5. Given $\frac{x}{2}-12=\frac{y}{4}+8$, and $\frac{x+y}{5}+\frac{x}{3}-8=\frac{-x}{4}+27$, to find x and y. *Anſ.* $x=60$ *and* $y=40$.

6. Given $a:b::x:y$, and $x^3-y^3=d$, to find d y.

$$Anſ.\ \left.\frac{da^3}{a^3-b^3}\right|^{\frac{1}{3}}=x,\ and\ \left.\frac{db^3}{a^3-b^3}\right|^{\frac{1}{3}}=y.$$

RULE III.

Let the given equations be multiplied or divided y ſuch numbers or quantities as will make the rm which contains one of the unknown quantities be the ſame in both equations; and then by lding or ſubtracting the equations, according as is quired, there will ariſe a new equation with only ne unknown quantity, as before.

EXAMPLES:

1. Given $\left\{\begin{array}{r}3x+5y=40\\ x+2y=14\end{array}\right\}$ to find x and y.

Firſt, multiply the 2d equation by 3,
and we ſhall have $3x+6y=42$,
Then, from this laſt equation ſubtract the firſt,
and it will give $6y-5y=42-40$, *or* $y=2$,
and therefore $x=14-2y=14-4=10$.

2. Given $\left\{\begin{array}{l}5x-3y=9\\2x+5y=16\end{array}\right\}$ to find x and y.

Let the 1ſt equation be multiplied by 2, and the 2d 5, and we ſhall have $10x-6y=18$

$$10x+25y=80$$

and if the former of theſe be ſubtracted from the latt

it will give $31y=62$, *or* $y=\frac{62}{31}=2$,

and conſequently $x=\frac{9+3y}{5}$, *by the firſt equation,*

or $x=\frac{9+6}{5}=\frac{15}{5}=3$.

Another method.

Multiply the firſt equation by 5, and the ſecond by and we ſhall have $25x-15y=45$

$$6x+15y=48$$

Now, let theſe two equations be added together,

and it will give $31x=93$, *or* $x=\frac{93}{31}=3$,

and conſequently $y=\frac{16-2x}{5}$, *by the 2d equation,*

or $y=\frac{16-6}{5}=\frac{10}{5}=2$ *as before.*

MISCELLANEOUS EXAMPLES.

1. Given $\frac{x+2}{3}+8y=31$, and $\frac{y+5}{4}+10x=1$ to find x and y. *Anſ.* $x=19$ *and* $y=$

2. Given $\frac{2x-y}{2}+14=18$, and $\frac{2y+x}{3}+16=1$ to find x and y. *Anſ* $x=5$ *and* $y=$

3. Given $\frac{2x+3y}{6}+\frac{x}{3}=8$, and $\frac{7y-3x}{2}-y=11$, to find x and y. *Anſ.* $x=6$ *and* $y=8$.

4. Given $ax+by=c$, and $dx+ey=f$, to find x and y. *Anſ.* $x=\frac{ce-bf}{ae-db}$ *and* $y=\frac{af-dc}{ae-db}$.

PROBLEM II.

To exterminate three unknown quantities, or to reduce the three ſimple equations, containing them, to a ſingle one.

RULE.

1. Let x, y, and z, be the three unknown quantities to be exterminated.

2. Find the value of x from each of the three given equations.

3. Compare the firſt value of x with the ſecond, and an equation will ariſe involving only y and z.

4. In like manner, compare the firſt value of x with the third, and another equation will ariſe involving only y and z.

5. Find the values of y and z from theſe two equations, according to the former rules, and x, y, and z will be exterminated as required.

Note, Any number of unknown quantities may be exterminated in nearly the ſame manner, but there are often much ſhorter methods for performing the operation, which will be beſt learnt from practice.

EXAMPLES:

1. Given $\left\{\begin{array}{l} x+y+z=29 \\ x+2y+3z=62 \\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=10 \end{array}\right\}$ to find x, y, and z.

From the first equation $x=29-y-z$.

From the second $x=62-2y-3z$.

From the third $x=20-\frac{2y}{3}-\frac{z}{2}$.

whence $29-y-z=62-2y-3z$,

and $29-y-z=20-\frac{2y}{3}-\frac{z}{2}$,

but, from the first of these equations, $y=62-29-2z$ $=33-2z$,

and from the second $y=27-\frac{3z}{2}$,

therefore $33-2z=27-\frac{3z}{2}$, *or* $z=12$,

and $y=62-29-2z=62-29-24=9$,

and $x=29-y-z=29-12-9=8$.

2. Given $\left\{\begin{array}{l} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62 \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47 \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38 \end{array}\right\}$ to find x, y, and z.

First, the given equations, cleared of fractions, become

$$12x+8y+6z=1488$$
$$20x+15y+12z=2820$$
$$30x+24y+20z=4560$$

Then, if the ſecond of theſe equations be ſubtracted from double the firſt, and three times the third from five times the ſecond, we ſhall have

$$4x+y=156$$
$$10x+3y=420$$

And again, if the ſecond of theſe be ſubtracted from three times the firſt, it will give

$$12x-10x=468-420, \textit{ or } x=\frac{48}{2}=24$$

Therefore $y=156-4x=60$, *and* $z=\frac{1488-8y-12x}{6}$ $=120$.

3. Given $x+y+z=31$, $x+y-z=25$, and $x-y-z=9$, to find x, y, and z.

Anſ. $x=20$, $y=8$, *and* $z=3$.

4. Given $x+y=a$, $x+z=b$, and $y+z=c$, to find x, y, and z.

5. Given $\left\{\begin{array}{l} ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p \end{array}\right\}$ to find x, y, and z.

A COLLECTION OF QUESTIONS

PRODUCING SIMPLE EQUATIONS.

1. To find two numbers, ſuch that their ſum ſhall be 40, and their difference 16.

Let x denote the leaſt of the two numbers required,
then will $x+16=$ *to the greater,*
and $x+x+16=40$ *by the queſtion,*
that is $2x=40-16=24$,

or $x=\frac{24}{2}=12=$ *leaſt number,*

and $x+16=12+16=28=$ *greater number requir*

2. What number is that whoſe $\frac{1}{3}$ part exceeds $\frac{1}{4}$ part by 16?

Let $x=$ *number required,*

then will its $\frac{1}{3}$ *part be* $\frac{x}{3}$, *and its* $\frac{1}{4}$ *part* $\frac{x}{4}$;

and therefore $\frac{x}{3}-\frac{x}{4}=16$ *by the queſtion,*

that is $x-\frac{3x}{4}=48$, *or* $4x-3x=192$;

whence $x=192$ *the number required.*

3. Divide 1000l. between A, B, and C, ſo t
A ſhall have 72l. more than B, and C 100l. m
than A.

Let $x=$ *B's ſhare of the given ſum,*
then will $x+72=$ *A's ſhare,*
and $x+172=$ *B's ſhare,*
And the ſum of all their ſhares $x+x+72+x+17$
or $3x+244=1000$ *by the queſtion,*
that is $3x=1000-244=756$,
or $x=\frac{756}{3}=252$l. $=$ *B's ſhare,*
and $x+72=252+72=324$l. $=$ *A's ſhare*
and $x+172=252+172=424$l. $=$ *C's ſh*

252l.
324l.
424l.

1000l. *the proof.*

5. A prize of 1000l. is to be divided between t
perſons, whoſe ſhares therein are in the proporti
of 7 to 9: required the ſhare of each.

Let $x=$ *firſt perſon's ſhare,*
then will $1000-x=$ *ſecond perſon's ſhare,*
and $x:\overline{1000-x}::7:9$, *by the queſtion,*
that is $9x=\overline{1000-x}\times 7=7000-7x$,
or $16x=7000$,
whence $x=\frac{7000}{16}=437$l. 10s. $=$ *1ſt ſhare,*
and $1000-x=1000-437$l. 10s. $=562$l. 10s. *2d ſhare.*

6. The paving of a ſquare at 2s. a yard, coſt as much as the incloſing it at 5s. a yard: required the ſide of the ſquare.

Let $x=$ *ſide of the ſquare ſought,*
then $4x=$ *yards of incloſure,*
and $x^2=$ *yards of pavement;*
whence $4x\times 5=20x=$ *price of incloſing,*
and $x^2\times 2=2x^2=$ *price of paving.*
But $2x^2=20x$ *by the queſtion,*
therefore $x^2=10x$, *and* $x=10=$ *length of the ſide required.*

7. A labourer engaged to ſerve for 40 days upon theſe conditions, that for every day he worked he ſhould receive 20 d. but for every day that he played, or was abſent, he was to forfeit 8 d. now at the end of the time he had to receive 1 l. 11 s. 8 d. the queſtion is, to find how many days he worked, and how many he was idle.

Let x *be the number of days he worked,*
then will $40-x$ *be the number of days he was idle,*
alſo $x\times 20=20x=$ *ſum earned,*
and $\overline{40-x}\times 8=320-8x=$ *ſum forfeited,*
whence $20x-\overline{320-8x}=380d.$ ($=1$ *l.* 11 *s.* 8*d.*) *by the queſtion, that is* $20x-320+8x=380$
or $28x=380+320=700$,

and $x=\frac{700}{28}=25=$ *number of days he w*

and $40-x=40-25=15=$ *number of d*
was idle.

8. Out of a caſk of wine, which had leaked $\frac{1}{3}$, 21 gallons were drawn; and then, being ga it appeared to be half full: How much did it l

Let it be ſuppoſed to have held x *gallons,*

Then it would have leaked $\frac{x}{3}$ *gallons,*

and conſequently there had been taken away 21 *gallons.*

But $21+\frac{x}{3}=\frac{x}{2}$ *by the queſtion,*

That is $63+x=\frac{3x}{2}$,

or $126+2x=3x$

hence $3x-2x=126$

or $x=126=$ *number of gallons required.*

9. What fraction is that, to the numera which if 1 be added, the value will be $\frac{1}{3}$; 1 be added to the denominator, its value will l

Let the fraction be repreſented by $\frac{x}{y}$,

Then will $\frac{x+1}{y}=\frac{1}{3}$,

And $\frac{x}{y+1}=\frac{1}{4}$,

or $3x+3=y$,

and $4x=y+1$,

hence $4x-3x-3=y+1-y$
that is $x-3=1$
or $x=4$, *and* $y=3x+3$
$=12+3=15$
So that $\frac{4}{15}=$ *fraction required.*

10. A market-woman bought in a certain number of eggs, at 2 a penny, and as many at three a penny, and ſold them all out again at the rate of 5 for 2 pence, and, by ſo doing, loſt 4 d. what number of eggs had ſhe?

Let $x=$ *number of eggs of each ſort.*

Then will $\frac{x}{2}=$ *price of the firſt ſort,*

and $\frac{x}{3}=$ *price of the ſecond ſort.*

But $5:2::2x$ (*the whole number of eggs*) $:\frac{4x}{5}$;

therefore $\frac{4x}{5}$ *price of both ſorts together, at five for 2d.*

And $\frac{x}{2}+\frac{x}{3}-\frac{4x}{5}=4$ *by the queſtion;*

that is $x+\frac{2x}{3}-\frac{8x}{5}=8$;

or $3x+2x-\frac{24x}{5}=24$;

or $15x+10x-24x=120$

Whence $x=120=$ *number of eggs of each ſort required.*

11. If A can do a piece of work alone in ten days, and B in thirteen; ſet them both about it together, in what time will it be finiſhed?

Let the time ſought be denoted by x,

Then 10 *days* : 1 *work* :: x *days* : $\frac{x}{10}$,

and 13 *days* : 1 *work* :: x *days* : $\frac{x}{13}$.

hence $\frac{x}{10} =$ *part done by A in x days*;

and $\frac{x}{13} =$ *part done by B in x days.*

Conſequently $\frac{x}{10} + \frac{x}{13} = 1$;

That is $\frac{13x}{10} + x = 13$, *or* $13x + 10x = 130$;

And therefore $23x = 130$, *or* $x = \frac{130}{23} = 5\frac{15}{23}$ *days, the time required.*

12. If one agent A, alone, can produce an effect e, in the time a, and another agent B, alone, in the time b; in what time will they both together produce the ſame effect?

Let the time ſought be denoted by x.

Then $a : e :: x : \frac{ex}{a} =$ *part of the effect produced by A*,

and $b : e :: x : \frac{ex}{b} =$ *part of the effect produced by B*,

whence $\frac{ex}{a} + \frac{ex}{b} = e$ *by the queſtion*;

or $\frac{x}{a} + \frac{x}{b} = 1$;

that is $x + \frac{ax}{b} = a$;

or $bx + ax = ba$;

and conſequently $x = \frac{ba}{b+a} =$ *time required.*

QUESTIONS FOR PRACTICE.

1. What two numbers are thoſe whoſe difference 7, and ſum 33. *Anſ.* 13 *and* 20.

2. To divide the number 75 into two ſuch parts, at three times the greater may exceed ſeven times e leſſer by 15. *Anſ.* 54 *and* 21.

3. In a mixture of wine and cyder, $\frac{1}{2}$ of the whole u 25 gallons was wine, and $\frac{1}{3}$ part *minus* 5 gallons is cyder: how many gallons were there of each?

Anſ. 85 *of wine, and* 35 *of cyder.*

4. A bill of 120 l. was paid in guineas and moi-res, and the number of pieces of both ſorts that re uſed was juſt 100; how many were there of :h? *Anſ.* 50 *of each.*

5. Two travellers ſet out at the ſame time from ndon and York, whoſe diſtance is 150 miles; e of them goes 8 miles a day, and the other 7; what time will they meet? *Anſ. in* 10 *days.*

6. At a certain election 375 perſons voted, and : candidate choſen had a majority of 91; how ny voted for each?

Anſ. 233 *for one, and* 142 *for the other.*

7. There is a fiſh whoſe tail weighs 9℔. his d weighs much as his tail and half his body, l his body weighs as much as his head and his l; what is the whole weight of the fiſh?

Anſ. 72℔.

8. What number is that from which if 5 be tracted, $\frac{2}{3}$ of the remainder will be 40. *Anſ.* 65.

9. A poſt is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and feet above the water; what is its whole length?

Anſ. 24 *feet.*

25. If three agents, A, B, and C, can produce the effects a, b, c, in the times e, f, g, reſpectively: in what time would they jointly produce the effect d?

Anſ. $\frac{\overline{e+f+g}\times d}{a+b+c}$ *time.*

26. If A and B together can perform a piece of work in 8 days; A and C together in 9 days; and B and C in 10 days: How many days will it take each perſon to perform the ſame work alone?

Anſ. A $14\frac{34}{49}$ *days, B* $17\frac{23}{41}$, *and C* $23\frac{7}{31}$.

QUADRATIC EQUATIONS.

A ſimple quadratic equation is that which involves the ſquare of the unknown quantity only.

An adfected quadratic equation is that which involves the ſquare of the unknown quantity, together with the product that ariſes from multiplying it by ſome known quantity.

Thus $ax^2=b$, *is a ſimple quadratic equation,*

And $ax^2+bx=c$, *is an adfected quadratic equation.*

The rule for a ſimple quadratic equation has been given already.

All adfected quadratic equations fall under the three following forms.

1. $x^2+ax=b$
2. $x^2-ax=b$
3. $x^2-ax=-b$.

The rule for finding the value of x, in each of theſe equations, is as follows:

RULE.*

1. Tranſpoſe all the terms that involve the unknown quantity to one ſide of the equation, and the known terms to the other ſide, and let them be ranged according to their dimenſions.

2. When the ſquare of the unknown quantity has any co-efficient prefixed to it, let all the reſt of the terms be divided by that co-efficient.

3. Add the ſquare of half the co-efficient of the ſecond term to both ſides of the equation, and that ſide which involves the unknown quantity will then be a complete ſquare.

4. Extract the ſquare root from both ſides of the equation, and the value of the unknown quantity will be determined, as required.

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[*] The ſquare root of any quantity may be either + or —, and therefore all quadratic equations admit of two ſolutions. Thus the ſquare root of $+n^2$ is $+n$ or $-n$; for $+n\times+n$ or $-n\times-n$ are each equal to $+n^2$. So, in the firſt form, where $x+\frac{a}{2}$ is found $=\sqrt{b+\frac{a^2}{4}}$ the root may be either $+\sqrt{b+\frac{a^2}{4}}$ or $-\sqrt{b+\frac{a^2}{4}}$, ſince either of them being multiplied by itſelf will produce $\sqrt{b+\frac{a^2}{4}}$. And this ambiguity is expreſſed by writing the uncertain ſign ± before $\sqrt{b+\frac{a^2}{4}}$; thus $x=\pm\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$.

Note, 1. The ſquare root of one ſide of the equation is always equal to the unknown quantity, with half the co-efficient of the ſecond term ſubjoined to it.

2. All equations wherein there are two terms involving the unknown quantity, and the index of the one is juſt double that of the other, are ſolved like quadratics, by completing the ſquare.

Thus, $x^4+ax^2=b$, *or* $x^n+ax^{\frac{n}{2}}=b$, *are the ſame as quadratics, and the value of the unknown quantity may be determined accordingly.*

In the firſt form, where $x=\pm\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$, the firſt value of x, viz. $x=+\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ is always affirmative; for ſince $\frac{a^2}{4}+b$ is greater than $\frac{a^2}{4}$, the greateſt ſquare muſt neceſſarily have the greateſt ſquare root; $\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ will, therefore, always be greater than $\sqrt{\frac{a^2}{4}}$, or its equal $\frac{a}{2}$; and conſequently $+\sqrt{b+\frac{a^2}{2}}-\frac{a}{2}$ will always be affirmative.

The ſecond value, viz. $x=-\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ will always be negative, becauſe it is compoſed of two negative terms. Therefore when $x^2+ax=b$, we ſhall have $x=+\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ for the affirmative value of x, and $x=-\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ for the negative value of x.

EXAMPLES:

1. Given $x^2+4x=140$ to find x.

Firſt, $x^2+4x+4=140+4=144$, *by completing e ſquare;*
then $\sqrt{x^2+4x+4}=\sqrt{144}$ *by extracting the root;*
or $x+2=12$,
and therefore $x=12-2=10$.

2. Given $x^2-6x+8=80$, to find x.

Firſt, $x^2-6x=80-8=72$ *by tranſpoſition;*
hen $x^2-6x+9=72+9=81$ *by completing the ſquare;*
And $x-3=\sqrt{81}=9$ *by extracting the root;*
Therefore $x=9+3=12$.

In the ſecond form, where $x=\pm\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ the rſt value, viz. $x=+\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ is always affirma- ve, ſince it is compoſed of two affirmative terms. The cond value, viz. $x=-\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ will always be gative; for ſince $b+\frac{a^2}{4}$ is greater than $\frac{a^2}{4}$, the ſquare root : $b+\frac{a^2}{4}$ $\left(\sqrt{b+\frac{a^2}{4}}\right)$ will be greater than $\sqrt{\frac{a^2}{4}}$, or its ual $\frac{a}{2}$; and conſequently $-\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ is always negative quantity. Therefore when $x^2-ax=b$, we ſhall ave $x=+\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ for the affirmative value of x, and $=-\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ for the negative value of x.

3. Given $2x^2+8x-20=70$, to find x.

Firſt, $2x^2+8x=70+20=90$ *by tranſpoſition.*
Then $x^2+4x=45$ *by dividing by* 2,
And $x^2+4x+4=49$ *by completing the ſquare;*
Whence $x+2=\sqrt{49}=7$ *by extracting the root.*
And conſequently $x=7-2=5$.

4. Given $3x^2-3x+6=5\frac{1}{3}$, to find x.

Here, $x^2-x+2=1\frac{7}{9}$ *by dividing by* 3,
And $x^2-x=1\frac{7}{9}-2$ *by tranſpoſition;*
Alſo $x^2-x+\frac{1}{4}=1\frac{7}{9}-2+\frac{1}{4}=\frac{1}{36}$ *by completing the ſqu.*
And $x-\frac{1}{2}=\sqrt{\frac{1}{36}}=\frac{1}{6}$ *by evolution;*
Therefore $x=\frac{1}{6}+\frac{1}{2}=\frac{2}{3}$.

In the third form, where $x=\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$, both the values of x will be poſitive, ſuppoſing $\frac{a^2}{4}$ is greater than b. For the firſt value, viz. $x=+\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$ is evidently affirmative, being compoſed of two affirmative terms. The ſecond value, viz. $x-\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$ is alſo affirmative; for ſince $\frac{a^2}{4}$ is greater than $\frac{a^2}{4}-b$, therefore $\sqrt{\frac{a^2}{4}}$ or $\frac{a}{2}$ is greater than $\sqrt{\frac{a^2}{4}-b}$, and conſequently $-\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$ will always be an affirmative quantity. Therefore, when $x^2-ax=-b$, we ſhall have $x=+\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$ for the affirmative value of x, and $-\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$ for the negative value of x.

iven $\frac{x^2}{2}-\frac{x}{3}+20\frac{1}{2}=42\frac{2}{3}$, to find x.

re $\frac{x^2}{2}-\frac{x}{3}=42\frac{2}{3}-20\frac{1}{2}=22\frac{1}{6}$ *by tranſpoſition,*

nd $x^2-\frac{2x}{3}=44\frac{1}{3}$ *by dividing by* $\frac{1}{2}$;

$x^2-\frac{2x}{3}+\frac{1}{9}=44\frac{1}{3}+\frac{1}{9}=44\frac{4}{9}$ *by completing the* re,

nd $x-\frac{1}{3}=\sqrt{44\frac{4}{9}}=6\frac{2}{3}$,
herefore $x=6\frac{2}{3}+\frac{1}{3}=7$.

iven $ax^2+bx=c$, to find x.

rſt, $x^2+\frac{b}{a}x=\frac{c}{a}$ *by diviſion;*

${}^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{c}{a}+\frac{b^2}{4a^2}$ *by completing the* re;

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this third form, if b be greater than $\frac{a^2}{4}$ the ſolution opoſed queſtion will be impoſſible. For ſince the ſquare uantity (whether that quantity be affirmative or nega- lways affirmative, the ſquare root of a negative quan- npoſſible, and cannot be aſſigned. But if b be greater , then $\frac{a^2}{4}-b$ is a negative quantity; and conſequently $-b$ is impoſſible, or only imaginary, when $\frac{a^2}{4}$ is b; and therefore in that caſe $x=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-b}$ is oſſible or imaginary.

And $x+\frac{b}{2a}=\sqrt{\frac{c}{a}+\frac{b^2}{4a^2}}=\sqrt{\frac{4ac+b^2}{4a^2}}$ *by evolution*;

Therefore $x=\pm\sqrt{\frac{4ac+b^2}{4a^2}}-\frac{b}{2a}$.

7. Given $ax^2-bx+c=d$, to find x.

Here, $ax^2-bx=d-c$ *by tranſpoſition,*

And $x^2-\frac{b}{a}x=\frac{d-c}{a}$ *by diviſion.*

Alſo $x^2-\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{d-c}{a}+\frac{b^2}{4a^2}$ *by completing the ſquare;*

And $x-\frac{b}{2a}=\pm\sqrt{\frac{d-c}{a}+\frac{b^2}{4a^2}}$ *by evolution;*

Therefore $x=\frac{b}{2a}\pm\sqrt{\frac{d-c}{a}+\frac{b^2}{4a^2}}$.

8. Given $x^4+2ax^2=b$, to find x.

Here, $x^4+2ax^2+a^2=b+a^2$ *by completing the ſquare,*

And $x^2+a=\sqrt{b+a^2}$ *by evolution;*

Whence $x^2=\sqrt{b+a}-a$,

And conſequently $x=\sqrt{:\sqrt{b+a}-a}$.

9. Given $ax^n-bx^{\frac{n}{2}}-c=-d$, to find x.

Firſt, $ax^n-bx^{\frac{n}{2}}=c-d$ *by tranſpoſition,*

And $x^n-\frac{b}{a}x^{\frac{n}{2}}=\frac{c-d}{a}$ *by diviſion,*

Alſo $x^n-\frac{b}{a}x^{\frac{n}{2}}+\frac{b^2}{4a^2}=\frac{c-d}{a}+\frac{b^2}{4a^2}$ *by completing the ſquare,*

And $x^{\frac{n}{2}}-\frac{b}{2a}=\sqrt{\frac{c-d}{a}+\frac{b^2}{2a^2}}$ *by evolution;*

Therefore $x^{\frac{n}{2}}=\frac{b}{2a}\pm\sqrt{\frac{c-d}{a}+\frac{b^2}{4a^2}}$,

And conſequently $x=\left.\overline{\frac{b}{2a}\pm\sqrt{\frac{c-d}{a}+\frac{b^2}{4a^2}}}\right|^{\frac{2}{n}}$

EXAMPLES FOR PRACTICE.

1. Given $x^2-8x+10=19$ to find x. *Anſ.* $x=9$.
2. Given $x^2-x-40=170$ to find x. *Anſ.* $x=15$.
3. Given $3x^2+2x-9=76$ to find x. *Anſ.* $x=5$.
4. Given $\frac{x^2}{2}-\frac{x}{3}+7\frac{1}{5}=20$ to find x. *Anſ.* $x=5.4093$, &c.
5. Given $x^2+x=a$ to find x.

Anſ. $x=\sqrt{a+\frac{1}{4}}-\frac{1}{2}$.

5. Given $x^2+ax-b=c$ to find x.

Anſ. $x=\sqrt{c+b+\frac{a^2}{4}}-\frac{a}{2}$.

. Given $x^2-ax=-b$ to find x.

Anſ. $x=\pm\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$.

. Given $\frac{ax^2}{b}-\frac{cx}{d}+\frac{e}{f}=\frac{g}{b}$ to find x.

Anſ. $x=\pm\sqrt{\frac{b^2c^2}{4a^2d^2}+\frac{bg}{ab}-\frac{be}{af}}+\frac{bc}{2ad}$.

. Given $2x^4-x^2+104=600$ to find x.

Anſ. $x=4$.

0. Given $3x^n-2x^{\frac{n}{2}}-\frac{p}{9}=\frac{r}{9}$ to find x.

Anſ. $x=\left.\overline{\pm\sqrt{\frac{r+p+3}{27}}+\frac{1}{3}}\right|^{\frac{2}{n}}$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To find two numbers whoſe difference is 8, and product 240.

Let $x=$ to the leaſt number,
then will $x+8=$ to the greater,
and $x\times\overline{x+8}=x^2+8x=240$ by the queſtion;
Whence $x^2+8x+16=240+16=256$ by completing the ſquare,
alſo $x+4=\sqrt{256}=16$ by evolution;
And therefore, $x=16-4=12=$ leſſer number, and $12+8=20=$ greater.

2. To divide the number 60 into two ſuch parts that their product may be 864.

Let $x=$ greater part,
then will $60-x=$ leſſer,
and $x\times\overline{60-x}=60x-x^2=864$ by the queſtion,
that is $x^2-60x=-864$;
Whence $x^2-60x+900=-864+900=36$ by completing the ſquare;
alſo $x-30=\sqrt{36}=6$ by extracting the root;
and therefore $x=6+30=36=$ greater part,
and $60-x=60-36=24=$ leſſer.

3. Given the ſum of two numbers $=10$ (a), and the ſum of their ſquares $=58$ (b); to find thoſe numbers.

Let $x=$ greater of the two numbers,
then will $a-x=$ leſſer;
And $x^2+\overline{a-x}|^2=2x^2+a^2-2ax=b$ by the queſtion,

or $x^2+\frac{a^2}{2}-ax=\frac{b}{2}$ *by diviſion,*

or $x^2-ax=\frac{b}{2}-\frac{a^2}{2}=\frac{b-a^2}{2}$ *by tranſpoſition.*

Whence $x^2-ax+\frac{a^2}{4}=\frac{b-a^2}{2}+\frac{a^2}{4}=\frac{2b-a^2}{4}$ *by completing the ſquare.*

Alſo $x-\frac{a}{2}=\sqrt{\frac{2b-a^2}{4}}$ *by extracting the root;*

And therefore $x=\pm\sqrt{\frac{2b-a^2}{4}}+\frac{a}{2}=$ *greater number,*

And $a-\overline{\frac{a}{2}\pm\sqrt{\frac{2b-a^2}{4}}}=\mp\sqrt{\frac{2b-a^2}{4}}+\frac{a}{2}=$ *leſſer.*

Hence theſe two theorems being put into numbers give 7 *and* 3, *for the numbers required.*

4. Sold a piece of cloth for 24l. and gained as much *per cent.* as the cloth coſt me; what was the price of the cloth?

Let $x=$ *pounds the cloth coſt,*
Then $24-x=$ *whole gain,*
But $100 : x :: x : \overline{24-x}$ *by the queſtion,*
Or $x^2=100\times\overline{24-x}=2400-100x$,
That is $x^2+100x=2400$,
Whence $x^2+100x+2500=2400+2500=4900$ *by completing the ſquare,*
And $x+50=\sqrt{4900}=70$ *by extraction of roots,*
Conſequently $x=70-50=20=$ *price of the cloth.*

5. A perſon bought a number of oxen for 80l. and if he had bought 4 more for the ſame money, he would have paid 1l. leſs for each; how many did he buy?

Suppoſe he bought x oxen,

Then $\frac{80}{x}=$ *price of each,*

And $\frac{80}{x+4}=$ *price of each if* $x+4$ *had coſt* 80l.

But $\frac{80}{x}=\frac{80}{x+4}+1$ *by the queſtion,*

Or $80=\frac{80x}{x+4}+x$,

Or $80x+320=80x+x^2+4x$,

That is $x^2+4x=320$,

Whence $x^2+4x+4=320+4=324$ *by completing the ſquare,*

And $x+2=\sqrt{324}=18$ *by evolution,*

Conſequently $x=18-2=16=$ *number of oxen required.*

6. What two numbers are thoſe, whoſe ſum, product, and difference of their ſquares are all equal to each other?

Let $x=$ *greater number,*

And $y=$ *leſſer,*

Then $\left\{\begin{array}{l} x+y=xy \\ x+y=x^2-y^2 \end{array}\right\}$ *by the queſtion,*

And $1=\frac{x^2-y^2}{x+y}=x-y$, *or* $x=y+1$ *from the 2d. equation.*

Alſo $\overline{y+1}+y=\overline{y+1}\times y$ *from the firſt equation,*

Or $2y+1=y^2+y$,

That is $y^2-y=1$,

Whence $y^2-y+\frac{1}{4}=1\frac{1}{4}$ *by completing the ſquare,*

Alſo $y-\frac{1}{2}=\sqrt{1\frac{1}{4}}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$ *by evolution,*

nſequently $y=\frac{\sqrt{5}}{2}+\frac{1}{2}=\frac{\sqrt{5}+1}{2}$, *and* $x=y+1$:
$\frac{\sqrt{5}+3}{2}$.

7. There are four numbers in arithmetical pro-
eſſion, whereof the product of the two extremes
45, and that of the means 77; what are the
ımbers?

Let $x=$ *leſſer extreme;*
and $y=$ *common difference,*
ben x, $x+y$, $x+2y$, $x+3y$ *will be the four numbers,*
d $\left\{\begin{array}{l} x\times\overline{x+3y}=x^2+3xy=45 \\ \overline{x+y}\times\overline{x+2y}=x^2+3xy+2y^2=77 \end{array}\right\}$ *by the queſtion,*

'hence $2y^2=77-45=32$, *and* $y^2=\frac{32}{2}=16$ *by ſub-*
traction and diviſion,
Or $y=\sqrt{16}=4$ *by evolution,*
berefore $x^2+3xy=x^2+12x=45$ *by the 1ſt. equation,*
'ſo $x^2+12x+36=45+36=81$ *by completing the ſquare,*
And $x+6=\sqrt{81}=9$ *by the extraction of roots,*
ınſequently $x=9-6=3$, *and the numbers are* 3, 7, 11, *and* 15.

8. To find three numbers in geometrical pro-
eſſion, whoſe ſum ſhall be 14, and the ſum of
eir ſquares 84.

Let x, y, *and* z *be the numbers ſought,*
Then $xz=y^2$ *by the nature of proportion,*
and $\left\{\begin{array}{l} x+y+z=14 \\ x^2+y^2+z^2=84 \end{array}\right\}$ *by the queſtion,*
But $x+z=14-y$ *by the 2d. equation,*

and $x^2+2xz+z^2=196-28y+y^2$ *by ſquaring both ſides,*
Or $x^2+z^2+2y^2=196-28y+y^2$ *by putting* $2y^2$ *for its equal* $2xz$,
That is $x^2+z^2+y^2=196-28y$ *by ſubtraction,*
or $196-28y=84$ *by equality,*
Hence $y=\frac{196-84}{28}=4$ *by tranſpoſition and diviſion.*
Again, $xz=y^2=16$, *or* $x=\frac{16}{z}$ *by the 1ſt equation,*
And $x+y+z=\frac{16}{z}+4+z=14$ *by the 2d equation,*
Or $16+4z+z^2=14z$, *or* $z^2-10z=-16$,
Whence $z^2-10z+25=25-16=9$ *by completing the ſquare,*
And $z-5=\sqrt{9}=3$, *or* $z=3+5=8$,
Conſequently $x=14-y-z=14-4-8=2$, *and the numbers are* 2, 4, 8.

9. The ſum (s) and the product (p) of any two numbers being given; to find the ſum of the ſquares, cubes, biquadrates, &c. of thoſe numbers.

Let the two numbers be denoted by x *and* y,
Then will $\left\{\begin{array}{r} x+y=s \\ xy=p \end{array}\right\}$ *by the queſtion.*
But $\overline{x+y}|^2=x^2+2xy+y^2=s^2$ *by involution;*
and $x^2+2xy+y^2-2xy=s^2-2p$ *by ſubtraction,*
that is $x^2+y^2=s^2-2p=$ *ſum of the ſquares.*
Again, $\overline{x^2+y^2}\times\overline{x+y}=\overline{s^2-2p}\times s$ *by multiplication,*
or $x^3+xy\times\overline{x+y}+y^3=s^3-2sp$,
or $x^3+sp+y^3=s^3-2sp$. *by ſubſtituting* sp *for its equal* $xy\times\overline{x+y}$;
and therefore $x^3+y^3=s^3-3sp=$ *ſum of the cubes.*

In like manner, $\overline{x^3+y^3}\times\overline{x+y}=\overline{s^3-3sp}\times s$ *by multiplication,*

or $x^4+xy\times\overline{x^2+y^2}+y^4=s^4-3s^2p$;

or $x^4+p\times\overline{s^2-2p}+y^4=s^4-3s^2p$ *by ſubſtituting* $p\times\overline{s^2-2p}$ *for its equal* $xy\times\overline{x^2+y^2}$;

And conſequently, $x^4+y^4=s^4-3s^2p-p\times\overline{s^2-2p}=s^4-4s^2p+2p^2=$ *ſum of the biquadrates, or fourth powers; and ſo on for any power whatever.*

10. The ſum (a) and the ſum of the ſquares (b) of four numbers, in geometrical progreſſion, being given; to find thoſe numbers.

Let x *and* y *denote the two means,*

then will $\frac{x^2}{y}$ *and* $\frac{y^2}{x}$ *be the two extremes, by the nature of proportion.*

Alſo, let the ſum of the two means $=s$, *and their product* $=p$.

And then will the ſum of the two extremes $=a-s$ *by the queſtion, and their product* $=p$, *by the nature of proportion.*

Hence $\left\{\begin{array}{l} x^2+y^2=s^2-2p \\ \frac{x^4}{y^4}+\frac{y^4}{x^2}=\overline{a-s}|^2-2p \end{array}\right\}$ *by the laſt problem,*

And $x^2+y^2+\frac{x^4}{y^2}+\frac{y^4}{x^2}=s^2+\overline{a-s}|^2-4p=b$ *by the queſtion.*

Again, $\frac{x^2}{y}+\frac{y^2}{x}=a-s$ *by the queſtion,*

or $x^3+y^3=xy\times\overline{a-s}=p\times\overline{a-s}$.

But $x^3+y^3=s^3-3sp$ *by the laſt problem;*

and therefore $p\times a-s=s^3-3sp$ *by equality;*

or $pa-ps+3ps=pa+2ps=s^3$,

or $p=\frac{s^3}{a+2s}$;

Whence $s^2+\overline{a-s}|^2-4p=s^2+\overline{a-s}|^2-\frac{4s^3}{a+2s}=b$, *by ſubſtitution,*

or $s^2+\frac{b}{a}s=\frac{a^2-b}{2}$ *by reduction.*

And $s=\sqrt{\frac{a^2-b}{2}+\frac{b^2}{4a^2}}-\frac{b}{2a}$ *by comp. the ſquare, and extracting the root.*

And from this value of s, all the reſt of the quantities p, x, and y may be readily determined.

QUESTIONS FOR PRACTICE.

1. What two numbers are thoſe whoſe ſum is 20, and their product 36? *Anſ.* 2 *and* 18.

2. To divide the number 60 into two ſuch parts, that their product may be to the ſum of their ſquares in the ratio of 2 to 5. *Anſ.* 20 *and* 40.

3. The difference of two numbers is 3, and the difference of their cubes is 117: what are thoſe numbers? *Anſ.* 2 *and* 5.

4. A company at a tavern had 8l. 15s. to pay for their reckoning; but, before the bill was ſettled, two of them ſneaked off, and then thoſe who remained had 10s. a-piece more to pay than before: how many were there in company? *Anſ.* 7.

6. A grazier bought as many ſheep as coſt him 60l. and, after reſerving 15 out of the number, he ſold the remainder for 54l. and gained 2s. a-head by them; how many ſheep did he buy? *Anſ.* 75.

7. There are two numbers whoſe difference is 15, and half their product is equal to the cube of the leſſer number; what are thoſe numbers? *Anſ.* 3 *and* 18.

. A perſon bought cloth for 33 l. 15 s. which he
again at 2 l. 8 s. *per* piece, and gained by the
gain as much one piece coſt him; required the
ber of pieces? *Anſ.* 15.

. What number is that, which when divided
the product of its two digits, the quotient is 3;
if 18 be added to it, the digits will be inverted?
Anſ. 24.

o. What two numbers are thoſe, whoſe ſum
ltiplied by the greater is equal to 77; and whoſe
erence multiplied by the leſſer is equal to 12?
Anſ. 4 *and* 7.

1. When will the hour, minute, and ſecond
ds of a clock be all together next after 12 o'clock?
Anſ. Only at 12 *o'clock.*

2. The ſum of two numbers is 8, and the ſum
eir cubes is 152; what are the numbers?
Anſ. 3 *and* 5.

3. The ſum of two numbers is 7, and the ſum
heir 4th powers is 641; what are the numbers?
Anſ. 2 *and* 5.

. The ſum of two numbers is 6, and the ſum
heir 5th powers is 1056; what are the num-
? *Anſ.* 2 *and* 4.

. The ſum of four numbers in arithmetical
reſſion is 56, and the ſum of their ſquares is
; what are the numbers?
Anſ. 8, 12, 16, *and* 20.

. To find four numbers in geometrical pro-
ion whoſe ſum is 15, and the ſum of their
res 85? *Anſ.* 1, 2, 4, *and* 8.

. Given $\overline{x^2-\frac{a^4}{x^2}}\Big|^{\frac{1}{2}}+\overline{a^2-\frac{a^4}{x^2}}\Big|^{\frac{1}{2}}=\frac{x^2}{a}$, to find
alue of x.

Anſ. $x=\overline{\tfrac{1}{2}a^2+\sqrt{\frac{5a^4}{4}}}\Big|^{\frac{1}{2}}$.

Of the NATURE and FORMATION of EQUATIONS in general.

All equations of ſuperior orders are generated by the multiplication of equations of inferior orders, involving the ſame unknown quantity.

Thus, *a quadratic equation* is formed by the multiplication of two ſimple equations.

A cubic equation is produced by the continued multiplication of three ſimple equations; or from one quadratic and one ſimple equation.

A biquadratic equation is generated by the continued multiplication of four ſimple equations, or of two quadratic equations; or of one cubic and one ſimple equation, &c.

For, ſuppoſe the unknown quantity to be x, and it values in any ſimple equation to be a, b, c, d, &c.

Then thoſe ſimple equations, by bringing all the terms to one ſide, will become $x-a=0$, $x-b=0$, $x-c=0$, $x-d=0$, &c.

And the product of any two of theſe, as $\overline{x-a}\times\overline{x-b}=0$, will give a *quadratic equation*, or one of two dimenſions.

The product of any three of them, as $\overline{x-a}\times\overline{x-b}\times\overline{x-c}=0$, will give a *cubic equation*, or one of three dimenſions.

The product of any four of them, as $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}=0$, will give a biquadratic equation, or one of four dimenſions, &c.

From hence it appears, that every equation has as many roots as it has ſimple equations that produce it, or as there are units in the higheſt dimenſion of the unknown quantity.

For if any of the values of x (a, b, c, *or* d,) be ſubſtituted in the place of x in this biquadratic equation $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}$, then will all the terms of that equation vaniſh, and the whole will be found equal to nothing. But as there are no other quantities, beſides theſe four, that ſubſtituted in the place of x will made the product vaniſh; therefore the equation cannot poſſibly have more than four roots, or admit of more than four ſolutions.

And after the ſame manner it appears, that no equation can have more roots than it contains dimenſions of the unknown quantity.

To make this ſtill plainer, by an example in numbers; ſuppoſe the equation to be reſolved be $x^4-10x^3+35x^2-50x+24=0$, and that you diſcover that this equation is the ſame with the product of $\overline{x-1}\times\overline{x-2}\times\overline{x-3}\times\overline{x-4}$.

Then you will certainly infer that the four values of x are 1, 2, 3, and 4; for any of theſe numbers being put for x will make that product, and conſequently $x^4-10x^3+35x^2-50x+24$, equal to nothing, according to the propoſed equation.

And it is certain that there can be no other values of x beſides theſe four; ſince if any other number be ſubſtituted for x in thoſe factors, there will none of them vaniſh, and therefore their product cannot be equal to nothing, according to the equation.

The roots of equations are either *poſitive* or *negative*, according as the roots of the ſimple equations from whence they are produced are poſitive or negative.

Thus, if you ſuppoſe $x=-a$, $x=b$, $x=-c$, and $x=d$, then will $x+a=0$, $x-b=0$, $x+c=0$, and $x-d=0$, and the equation $\overline{x+a}\times\overline{x-b}\times\overline{x+c}\times\overline{x-d}=0$ will have its roots $-a$, $+b$, $-c$, $+d$.

The *ſigns*, and *co-efficients* of equations will be beſt underſtood by conſidering the following table; where the ſimple equations $x-a$, $x-b$, &c. are multiplied continually together, and produce, ſucceſſively, the higher equations.

$$\begin{array}{l}
x-a=0\\
x-b=0\\
\hline
\left.\begin{array}{l} x^2-ax \\ \quad -bx+ba \end{array}\right\} =0,\textit{ a quadratic,}\\
\hline
\times\, x-c=0\\
\hline
x^3\left.\begin{array}{r} -a \\ -b \\ -c \end{array}\right\} x^2 \left.\begin{array}{l} +ab \\ +ac \\ +bc \end{array}\right\} x-abc=0,\textit{ a cubic,}\\
\hline
\times\, x-d=0\\
\hline
x^4\left.\begin{array}{r} -a \\ -b \\ -c \\ -d \end{array}\right\} x^3 \left.\begin{array}{l} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{array}\right\} x^2 \left.\begin{array}{l} -abc \\ -abd \\ -acd \\ -bcd \end{array}\right\} x+abcd=0,\textit{ a biquadratic.}\\
\hline
\end{array}$$

&c.

From the inſpection of theſe equations it is plain, that the co-efficient of the firſt term is *unity*.

The co-efficient of the ſecond term is *the ſum of all the roots* (a, b, c, d,) *with contrary ſigns*.

The co-efficient of the third term is *equal to the ſum of the rectangles of thoſe roots, or of all the products that can poſſibly ariſe by combining them, two and two.*

The co-efficient of the fourth term *is equal to the ſum of all the products that can poſſibly ariſe by combin-*

g them, three and three; and ſo on for any other -efficient whatever.

The laſt term *is always equal to the product of all e roots with contrary ſigns*; and this reaſoning will old, whether the roots be poſitive or negative.

It likewiſe appears, from inſpection, that the ſigns all the terms of any equation in the table are ternately + and −.

The firſt term is always ſome pure power of *x*, and poſitive.

The ſecond term is ſome power of *x*, multiplied y the quantities −*a*, −*b*, −*c*, &c. and ſince theſe uantities are all negative, it follows that the term tſelf muſt be negative alſo.

The third term has the product of any two of heſe quantities (−*a*, −*b*, −*c*,) for its co-efficient, and is therefore poſitive; ſince −×−, as well as +×+, gives +, or an affirmative quantity.

For the ſame reaſon, the next co-efficient, which is formed of the products of any three of theſe negative quantities, muſt be negative; and the next following co-efficient, being made up of the products of any four of the ſaid negative quantities, muſt be poſitive, and ſo on.

And from this reaſoning it plainly appears that *when all the roots are poſitive, the ſigns are plus and inus alternately*.

But if the roots are all negative ($x=-a$, $x=-b$, $=-c$, $x=-d$) then $\overline{x+a}\times\overline{x+b}\times\overline{x+c}\times\overline{x+d}=0$, will expreſs the equation to be produced; and all he terms will, plainly, be poſitive.

So that, *when all the roots of an equation are negaive, it is plain that there can be no changes in the ſigns f the terms of that equation*.

And, in general, there will be as many poſitive oots in any equation, as there are changes in the ſigns of the terms of that equation, from + to −, or from − to +; and all the reſt of the roots will be negative.

From this rule it follows that, in quadratic equations, the two roots may be either both poſitive, or both negative, or one negative and one affirmative.

Thus, in the equation, $x^{2}\begin{matrix}-ax\\-bx\end{matrix}+ab=o$, $(\overline{x-a}\times\overline{x-b})$ there are two changes of the ſigns, and therefore the roots are both poſitive.

In the equation $x^{2}\begin{matrix}+ax\\+bx\end{matrix}+ab=o$, $(\overline{x+a}\times\overline{x+b})$ there is no change of the ſigns, and conſequently they are both negative.

And, in this equation, $x^{2}\begin{matrix}-ax\\-bx\end{matrix}+ab=o$, $(\overline{x-a}\times\overline{x+b})$ one of the roots will be affirmative and one negative; for ſince the firſt term is poſitive and the laſt negative, there can be but one change in the ſigns, whether the ſecond term be + or −.

In cubic equations, the roots may be all poſitive, or all negative; or two of them may be negative and one poſitive, or one negative and two poſitive.

Thus, in the equation $\overline{x-a}\times\overline{x-b}\times\overline{x-c}=o$, the ſigns will be alternately + and −; and, as the number of changes is three, the roots muſt be all poſitive.

In the equation $\overline{x+a}\times\overline{x+b}\times\overline{x+c}=o$, where there are no changes of the ſigns, the roots muſt be all negative.

In the equation $x^{3}-\overline{a-b+c}\,x^{2}+\overline{ab-ac-bc}\,x+abc=o$, $(\overline{x-a}\times\overline{x-b}\times\overline{x+c})$ the number of changes will be two, and conſequently two of the roots will be affirmative and one negative.

For if $a+b$ is greater than c, the ſecond term muſt be negative, its co-efficient being $-a$, $-b$, $+c$;

and if $a+b$ is leſs than c, the third term muſt be negative, its co-efficient $+ab-ac-bc$ ($ab-c\times\overline{a+b}$) being in that caſe negative.

In the equation $x^3+\overline{a+b-c}x^2+\overline{ab-ac-bc}x-abc=0$, there can be only one change of the ſigns, and therefore one of the roots is poſitive, and the other two negative.

For if $a+b$ is leſs than c then the ſecond term is negative, and the third muſt be negative alſo; and if $a+b$ is greater than c the ſecond term will be poſitive, and there can be but one change in the other two terms, whatever may be their ſigns.

And, in the ſame manner, this reaſoning may be extended to equations of higher dimenſions, and therefore the rule will be found to be true in all kinds of equations whatever.

PROBLEM I.

To increaſe or diminiſh the roots of an equation by any given quantity.

RULE.

1. Take ſome new letter, and connect it with the given quantity by the ſigns — or +, according as it is required to be increaſed or diminiſhed.

2. Subſtitute the powers of this quantity in the equation, inſtead of the powers of the unknown letter, and there will ariſe a new equation, whoſe roots will be augmented or diminiſhed as required.

EXAMPLES:

1. Let the quadratic equation $x^2-8x+15=0$, be given; it is required to increaſe its roots by 7.

$$\begin{array}{l} \textit{Suppoſe } x=y-7, \\ \textit{Then } x^2=y^2-14y+49 \\ \qquad 8x=\;+\;8y-56 \\ \qquad 15=\qquad\;\;+15 \\ \hline \qquad\quad y^2-\;6y+\;8=0=\textit{ equal to the equation required}^{*}. \end{array}$$

2. Let $x^3-px^2+qx-r=0$, be the equation given; it is required to diminiſh the roots by the quantity e.

$$\textit{Suppoſe } x=y+e;$$

$$\left.\begin{array}{l} \textit{Then } x^3=y^3+3ey^2+3e^2y+e^3 \\ \quad -px^2=\;-py^2-2pey-pe^2 \\ \quad +qx=\qquad\quad +qy\;+qe \\ \quad -r=\qquad\qquad\quad -r \end{array}\right\}=0, \textit{ the new equation required}\dagger.$$

3. Let $x^3+x^2-10x+8=0$ be given, and let its roots be increaſed by 4.

---

* For in the former equation $x^2+8x+15=0$, the roots are -3 and -5, and in the equation $y^2-6y+8=0$, the roots are 2 and 4; and therefore the difference is 7, as required.

† The laſt term of this transformed equation, is the ſame as the equation given, having e in the place of y.

Suppoſe $x=y-4$;

$$\begin{array}{rl}
\textit{Then } x^3 = & y^3-12y^2+48y-64 \\
+\ x^2 = & +\ y^2-\ 8y+16 \\
-10x = & \qquad -10y+40 \\
+\ 8 = & \qquad\qquad +\ 8 \\
\hline
\end{array}$$

$y^3-11y^2+30y=0$, *or* $y^2-11y+30=0$, *the equation required* ‡.

PROBLEM II.

To take away the ſecond term from any equation.

RULE.

1. Divide the co-efficient of the ſecond term by the index of the higheſt power of the unknown quantity.

2. Annex the quotient, with its ſign changed, to ſome new letter, and this being ſubſtituted for its equal in the given equation, will deſtroy the ſecond term, as required.

---

‡ In this example the given equation is reduced to a quadratic, and in the preſent caſe, as well as in all others where the laſt term vaniſhes, the number aſſumed (-4) is one of the roots of the propoſed equation.

The affirmative roots of an equation are changed into negative ones of the ſame value, and the negative roots into affirmative ones, by only changing the ſigns of the terms alternately, beginning with the ſecond.

EXAMPLES:

1. Let the quadratic equation $x^2-8x+15=0$ be given; it is required to take away its ſecond term.

Suppoſe $x=y+4(y+\frac{8}{2})$;

Then $x^2=y^2+8y+16$

$-8x=-8y-32$

$+15=+15$

$y^2-1=o=$ *equation required**.

2. Let the equation $x^3-9x^2+26x-34=0$ be given; it is required to exterminate its ſecond term.

Suppoſe $x=y+3(y+\frac{9}{3})$;

Then $x^3=y^3+9y^2+27y+27$

$-9x^2=-9y^2-54y-81$

$+26x=+26y+78$

$-34=-34$

$y^3-y-10=o=$ *equation required.*

Thus the roots of the equation $x^4-x^3-19x^2+49x-30=0$, are $+1$, $+2$, $+3$, -5; but by changing only the ſecond and fourth terms, the equation becomes $x^4+x^3-19x^2-49x-30=o$, and the roots are -1, -2, -3, $+4$.

All the roots of an equation may alſo be made affirmative or negative, by increaſing or diminiſhing each of them by ſome known quantity.

* From this example it appears, that any quadratic equation may be reſolved without completing the ſquare, by only taking away the ſecond term; for ſince $y^2=1$, or $y=\sqrt{1}=1$, we ſhall have $x=y+4=1+4=5$, the root required.

3. Let $x^4+8x^3-5x^2+10x-4=0$ be given, to exterminate the ſecond term.

Suppoſe $x=y-2(y-\frac{8}{4})$;

$$\begin{array}{rl}
\textit{Then } x^4= & y^4-8y^3+24y^2-32y+16 \\
+\ 8x^3= & +8y^3-48y^2+96y-64 \\
-\ 5x^2= & \qquad\quad -5y^2+20y-20 \\
+10x= & \qquad\qquad\qquad +10y-20 \\
-\ 4= & \qquad\qquad\qquad\qquad\ \ -4 \\
\hline
 & y^4-29y^2+94y-92=0=\textit{equation required.}
\end{array}$$

4. Let $x^4-px^3+qx^2-rx+s=0$ be given, to exterminate the ſecond term.

Suppoſe $x=y+\frac{p}{4}$;

$$\begin{array}{rl}
\textit{Then } x^4= & y^4+py^3+\frac{3p^2y^2}{8}+\frac{p^3y}{16}+\frac{p^4}{256} \\
-px^3= & -py^3-p^2y^2-\frac{3p^3y}{16}-\frac{p^4}{16} \\
+qx^2= & \qquad\quad +qy^2+\frac{pqy}{2}+\frac{qp^2}{16} \\
-rx= & \qquad\qquad\qquad -ry-\frac{rp}{4} \\
+s= & \qquad\qquad\qquad\qquad +s \\
\hline
\end{array}$$

$$y^4+\frac{8q-5p^2}{8}y^2+\frac{8pq-2p^3-16r}{16}$$

$$y+\frac{-15p^4+16qp^2-64rp+256s}{256}=0=\textit{equation requir.}$$

PROBLEM III.

To find whether ſome, or all the roots of an equation be rational; and, if ſo, what they are.

RULE.*

1. Find all the diviſors of the laſt term, and ſubſtitute them one by one for the unknown quantity.

2. Then, if the poſitive and negative terms deſtroy each other, the diviſor, ſo ſubſtituted, will be one of the roots of the equation.

3. But, if none of the diviſors ſucceed, the roots are, for the general part, either irrational or impoſſible.

Note, When the diviſors of the laſt term are too numerous, they may be diminiſhed by changing the equation into another, whoſe roots are augmented or decreaſed by an unit, or ſome other known quantity.

EXAMPLES:

1. Let $x^3-4x^2-7x+10=0$ be the equation propoſed.

Then the diviſors of (10) *the laſt term will be* $+1$, -1, $+2$, -2, $+5$, -5, $+10$, -10.

---

* Since the laſt term, in any equation, is always equal to a multiple of all the roots in that equation, thoſe roots muſt, therefore, neceſſarily be found in the number of its diviſors.

And theſe being ſubſtituted ſucceſſively inſtead of x, *will give*

$$\begin{aligned} 1-4-7+10&=0\\ -1-4+7+10&=12\\ 8-16-14+10&=-12\\ -8-16+14+10&=0\\ 125-100-35+10&=0 \end{aligned}$$

Therefore $+1$ -2 *and* $+5$ *are the three roots of the equation roquired.*

2. Let $y^4-4y^3-8y+32=0$, be the equation propoſed.

1. *Change it into another, the number of whoſe diviſors ſhall be leſs; thus:*

Suppoſe $y=x+1$;

$$\begin{aligned} \textit{Then}\quad y^4&=x^4+4x^3+6x^2+4x+1\\ -4y^3&=-4x^3-12x^2-12x-4\\ -8y&=-8x-8\\ +32&=+32 \end{aligned}$$

$$x^4-6x^2-16x+21=0=\textit{ new equation.}$$

2. *The diviſors of the laſt term* (21) *of this new equation are*

1, −1, +3, −3, +7, −7, +21, −21.

And if theſe be ſubſtituted ſucceſſively inſtead of x, *we ſhall hove*

$$\begin{aligned} 1-6-16+21&=0\\ 1-6+16+21&=32\\ 81-54-48+21&=0\\ 81-54+48+21&=96 \end{aligned}$$

&c. where none of the others ſucceed.

So that 1 *and* 3 *are the only rational roots, the other two being impoſſible.*

3. Let $x^3+3ax^2-4a^2x-12a^3=0$, be the equation propoſed.

Here the numeral diviſors of the laſt term ($12a^3$) *are*

1, −1, +2, −2, +3, −3, +4, −4, +6, −6, +12, −12.

And by ſubſtituting theſe ſucceſſively inſtead of x, we ſhall have

$$\begin{aligned} 1+3-4-12&=-12\\ -1+3+4-12&=-6\\ 8+12-8-12&=0\\ -8+12+8-12&=0\\ 27+27-12-12&=30\\ -27+27+12-12&=0 \end{aligned}$$

Therefore the three roots are $2a$, $-2a$ *and* $3a$.

PROBLEM IV.

To diſcover the roots of equations by SIR ISAAC NEWTON's *method of diviſors.*

RULE.

1. Inſtead of the unknown quantity, ſubſtitute ſucceſſively three, or more, terms of the arithmetical progreſſion 2, 1, 0, —1, —2.

2. Collect all the terms of the equation into one ſum; and place them, together with their diviſors, in a perpendicular line, right againſt the correſponding terms of the progreſſion 2, 1, 0, —1, —2.

3. Seek amongſt the diviſors for an arithmetical progreſſion whoſe terms correſpond with the order of the terms 2, 1, 0, —1, —2, and whoſe common difference is either an unit, or ſome diviſor of the co-efficient of the higheſt power of the unknown quantity in the given equation.

4. Divide that term of the progreſſion, thus found, which ſtands againſt the term 0 in the firſt progreſſion by the ratio or common difference.

5. To the quotient, laſt found, prefix the ſign + or —, according as the progreſſion is increaſing or

decreaſing, and this number being ſubſtituted for the unknown quantity, will give one of the roots of the equation.

EXAMPLES:

1. Let $x^3-x^2-10x+6=0$, be the equation propoſed.

Then, by ſubſtituting ſucceſſively the terms of the progreſſion 2, 1, 0, —1, *inſtead of x, the work will ſtand as follows:*

1ſt prog.	*reſults.*	*diviſors.*	*2d prog.*
2	—10	1 . 2 . 5 . 10	5
1	— 4	1 . 2 . 4	4
0	+ 6	1 . 2 . 3 . 6	3
—1	+ 14	1 . 2 . 7 . 14	2

And —3, *the term ſtanding againſt* 0, *being ſubſtituted for x, gives* $-27-9+30+6=0$; *and therefore* —3 *is a root of the equation.*

2. Let $2x^3-5x^2+4x-10=0$, *be the equation propoſed.*

Then, by ſubſtituting ſucceſſively the terms of the progreſſion 2, 1, 0, —1, —2, *inſtead of x, the work will ſtand as follows:*

1ſt prog.	*reſults.*	*diviſors.*	*2d. prog.*
2	— 6	1 . 2 . 3 . 6	1
1	— 9	1 . 3 . 9	3
0	—10	1 . 2 . 5 . 10	5
—1	—21	1 . 3 . 7 . 21	7
—2	—54	1 . 2 . 3 . 6 . 9	9

Here 5, *the term ſtanding againſt* 0, *being divided by* 2, *the common difference, gives* $2\frac{1}{2}$; *and this being ſubſtituted for x, gives* $31\frac{1}{4}-31\frac{1}{4}+10-10=0$; *and therefore* $2\frac{1}{2}$ *is a root of the equation.*

3. Let $x^4+x^3-29x^2-9x+180=0$, be the equation propoſed.

Then, by ſubſtituting as before, the work will ſtand as follows:

1ſt pro.	*reſults.*	*diviſors.*	*progreſſions.*			
2	70	1.2.5.7.10.14 &c.	1	2	5	7
1	144	1.2.3.4. 6. 8 &c.	2	3	4	6
0	180	1.2.3.4. 5. 6 &c.	3	4	3	5
—1	160	1.2.4.5. 8.10 &c.	4	5	2	4
—2	90	1.2.3.5. 6. 9 &c.	5	6	1	3

So that here are four progreſſions, and the numbers 3, 4, —3, and —5, being each ſubſtituted for x, make the whole equation vaniſh, and are therefore the roots required.

PROBLEM V.

To find the roots of cubic equations, according to the method of CARDAN.

RULE.*

1. Take away the ſecond term of the equation, by problem 2d, and it will be reduced to this form $x^3\pm ax=\pm b$.

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* The rule, from whence this method is derived, is $x=\sqrt[3]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}+\sqrt[3]{\frac{1}{2}b-\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$, and the inveſtigation of it is as follows:

Let the equation, whoſe root is required, be $x^3+ax=b$.

Put $y+z=x$, and $3yz=-a$.

Then, by ſubſtituting theſe values in the given equation, we ſhall have, $y^3+3y^2z+3yz^2+z^3+a\times\overline{y+z}=y^3+z^3+3yz\times\overline{y+z}+a\times\overline{y+z}=y^3+z^3-a\times\overline{y+z}+a\times\overline{y+z}=y^3+z^3=b$.

2. Subſtitute the values of a and b, with their proper ſigns, in the following expreſſion, and it will give the root required. Thus:

$$\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}}=x=$$

root required.

Note, When a is negative, and $\frac{1}{27}a^3$ is greater than $\frac{1}{4}b^2$, the equation, by this rule, is generally impoſſible.

And if, from the ſquare of this laſt equation, there be taken 4 times the cube of the equation $yz=-\frac{1}{3}a$, we ſhall have $y^6-2y^3z^3+z^6=b+\frac{4}{27}a^3$, or $y^3-z^3=\sqrt{b+\frac{4}{27}a^3}$.

But the ſum of this equation and $y^3+z^3=b$, is $2y^3=b+\sqrt{b+\frac{4}{27}a^3}$; and their difference is $2z^3=b-\sqrt{b+\frac{4}{27}a^3}$; and therefore $y=\sqrt[3]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$, and $z=\sqrt[3]{\frac{1}{2}b-\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$.

And from hence it appears, that $y+z$, or its equal x, is $=\sqrt[3]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}+\sqrt[3]{\frac{1}{2}b-\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$, which is the theorem.

Or, ſince z is $=-\frac{\frac{1}{3}a}{y}$, it will be $y+z=x=$

$$\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}},$$ *the ſame as the rule.*

This method of ſolving cubic equations is uſually aſcribed to *Cardan*; but the invention is, undoubtedly, not his.—The authors of it were *Scipio Ferreus* and *Nicholas Tartalea,* who diſcovered it about the ſame time, independently of each other, as is proved by *M. de Montucla* dans ſon *Hiſtoire des Mathématiques.*

EXAMPLES:

1. Let $y^3+3y^2+9y=13$, be the equation propoſed; it is required to find the value of y.

1. *In order to deſtroy the ſecond term, let* $y=x-1$; *then*

$$\begin{aligned} y^3 &= x^3-3x^2+3x-1 \\ 3y^2 &= \qquad +3x^2-6x+3 \\ 9y &= \qquad\qquad\quad +9x-9 \\ \hline & \quad x^3+6x-7=13 \\ & \text{or } x^3+6x=20. \end{aligned}$$

2. *For a put* 6, *and for b* 20, *and we ſhall have*

$$x=\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}}=$$

$$\sqrt[3]{10+\sqrt{100+8}}-\frac{2}{\sqrt[3]{10+\sqrt{100+8}}}=$$

$$\sqrt[3]{10+10.3923}-\frac{2}{\sqrt[3]{10+10.3923}}=\sqrt[3]{20.3923}-$$

$$\frac{2}{\sqrt[3]{20.3923}}=2.732-\frac{2}{2.732}=2.732-.732=2;$$ *that is* $x=2$, *and conſequently* $y=1=$ *root required.*

2. Given $x^3-6x=-9$, to find the value of x.

Here $a=-6$, *and* $6=-9$; *and therefore we ſhall have*

$$x=\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}}=$$

$$\sqrt[3]{-4\tfrac{1}{2}+\sqrt{20\tfrac{1}{4}-8}}-\frac{-2}{\sqrt[3]{-4\tfrac{1}{2}+\sqrt{20\tfrac{1}{4}-8}}}=$$

$$\sqrt[3]{-4\tfrac{1}{2}+3\tfrac{1}{2}}-\frac{-2}{\sqrt[3]{-4\tfrac{1}{2}+3\tfrac{1}{2}}}=\sqrt[3]{-1}-\frac{-2}{\sqrt[3]{-1}}=-1$$

$-\frac{-2}{-1}=-1-2=-3$; *that is* $x=-3=$ *root required.*

EXAMPLES FOR PRACTICE.

1. Given $x^3-6x^2+10x-8=0$, to find x. *Anſ.* $x=4$.
2. Given $x^3+30x=117$, to find x. *Anſ.* $x=3$.
3. Given $y^3-36y=91$, to find y. *Anſ.* $y=7$.
4. Given $x^3+6x=30\sqrt{3}$, to find x. *Anſ.* $x=2\sqrt{3}$.
5. Given $y^3+24y=250$, to find y. *Anſ.* $y=5.05$.
6. Given $y^6-3y^4-2y^2-8=0$, to find y. *Anſ.* $y=2$.

PROBLEM VI.

To find the roots of biquadratic equations, according to the method of DES CARTES.

RULE.*

1. Take away the ſecond term of the equation by problem 2, and it will be reduced to the form $x^4+qx^2+rx+s=0$.

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* *Inveſtigation of the rule.* Let the given equation $x^4+qx^2+rx+s=0$, be equal to the product of the two quadratic equations $x^2+ex+f=0$, and $x^2-ex+g=0=x^4+f+g-e^2\,x^2+eg-ef\,x+fg=0$.

Then, by equating the homologous terms, we ſhall have $f+g-e^2=q$, $eg-ef=r$, and $fg=s$; and therefore $f=$

3. *To find the value of p, by* CARDAN's *rule for cubic equations.*

$$\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}}=$$

$$\sqrt[3]{288+\sqrt{\overline{288}|^2-\overline{32}|^3}}-\frac{-32}{\sqrt[3]{288+\sqrt{\overline{288}|^2-\overline{32}|^3}}}=$$

$12=p$; *and therefore* $y=16$, *or* $\sqrt{y}=4$, $f=\frac{-6}{2}+\frac{16}{2}+\frac{16}{8}=7$, *and* $g=\frac{-6}{2}+\frac{16}{2}-\frac{16}{8}=3$.

4. *To find the roots of the two quadratic equations* $x^2+ex+f=o$, *and* $x^2-ex+g=o$.

---

1. $2A+\tfrac{1}{4}a^2-B^2=b$, or $2A+\tfrac{1}{4}a^2-b=B^2$;
2. $aA-2BC=c$, or $aA-c=2BC$;
3. $A^2-C^2=d$, or $A^2-d=C^2$.

Let now the firſt and laſt of theſe equations be multiplied together, and the product will, evidently, be equal to $\tfrac{1}{4}$ of the ſquare of the ſecond; that is $2A^3+\overline{\tfrac{1}{4}a^2-b}\times A^2-2dA-\overline{\tfrac{1}{4}a^2-b}\times d\,(=B^2C^2)=\tfrac{1}{4}\times\overline{a^2A^2-2acA+c^2}$.

Whence, by denoting the given quantities $\tfrac{1}{4}ac-d$, and $\tfrac{1}{4}c^2+d\times\overline{\tfrac{1}{4}a^2-b}$ by k and l, reſpectively, there will ariſe this cubic equation, $A^3-\tfrac{1}{2}bA^2+kA-\tfrac{1}{2}l=o$; by means whereof, the value of A may be determined; and therefore, from the preceding equations, both B and C will alſo be known, B being found from thence $=\sqrt{2A+\tfrac{1}{4}a^2-b}$, and C $=\frac{aA-c}{2B}$.

The ſeveral values of A, B, and C being thus found, that of x will alſo be readily obtained: for $\overline{x^2+\tfrac{1}{2}ax+A}|^2-\overline{Bx+C}|^2$ being univerſally, in all circumſtances of x, equal to $x^4+ax^3+bx^2+cx+d$, it is evident, that, when the value of x is taken ſuch that the latter of theſe expreſſions becomes equal to nothing, the former muſt likewiſe be equal to nothing; and conſequently $\overline{x^2+\tfrac{1}{2}ax+A}|^2=\overline{Bx+C}|^2$.

$$x^2+ex+f=x^2+4x+7=0$$
$$x^2-ex+g=x^2-4x+3=0$$

In the firſt of theſe $x=-2+\sqrt{-3}$, *or* $-2-\sqrt{-3}$,
And in the ſecond $x=3$, *and* 1.

Therefore 3, 1, $-2+\sqrt{-3}$, *and* $-2-\sqrt{-3}$ *are the four roots of the equation* $x^4-6x^2-16x+21=0$.

And if unity be added to each of them, we ſhall have 4, 2, $-1+\sqrt{-3}$, *and* $-1-\sqrt{-3}$ *for the roots of* $z^4-4z^3-8z+32=0$, *the equation propoſed; the two laſt of which are impoſſible.*

2. Given $x^4+2x^3-7x^2-8x+12=0$, to find the values of x. *Anſ.* $x=1$, 2, -3 *and* -2.

3. Given $x^4-25x^2+60x-36=0$, to find the values of x. *Anſ.* 1, 2, 3 *and* -6.

4. Given $y^4-8y^3+14y^2+4y-8=0$, to find the value of y.

Anſ. $y=3+\sqrt{5}$, $3-\sqrt{5}$, $1+\sqrt{3}$, *and* $1-\sqrt{3}$.

And therefore, by extracting the ſquare root of both ſides of the equation, we ſhall have $x^2+\frac{1}{2}ax+A=\pm Bx\pm C$; or $x=\pm\frac{1}{2}B-\frac{1}{4}a\pm\overline{\overline{\tfrac{1}{4}a\pm\tfrac{1}{2}B}^2\pm C-A}^{\frac{1}{2}}=\pm\frac{1}{2}B-\frac{1}{4}a\pm\overline{\tfrac{1}{16}a^2\mp\tfrac{1}{4}aB+\tfrac{1}{4}B^2\pm C-A}^{\frac{1}{2}}$; which exhibits all the different roots of the given equation, according to the variation of the ſigns.

This method will be found to have many advantages over that given above. In the firſt place, there is no neceſſity of being at the trouble of exterminating the ſecond term of the equation, in order to prepare it for a ſolution: ſecondly, the equation $A^3-\frac{1}{2}bA^2+kA-\frac{1}{2}l=0$, here brought out, is of a more ſimple form than that derived from the former method: and thirdly, the value of A in this equation, will always be *commenſurate* and *rational*; not only when all the roots of the given equation are *commenſurate*, but alſo when they are *irrational*, and even *impoſſible.*

Example. *Let there be given* $x^4+12x-17=0$, *to find the values of* x.

5. Given $x^4 + 12x - 17 = 0$, to find the values of x.

Anſ. $x = \frac{1}{2}\sqrt{2} \pm \sqrt{-3\sqrt{2} - \frac{1}{2}}$, *and* $-\frac{1}{2}\sqrt{2} \pm \sqrt{3\sqrt{2} - \frac{1}{2}}$.

6. Given $x^4 - 6x^3 - 58x^2 - 114x - 11 = 0$, to find the values of x.

$$\textit{Anſ.}\ x = \pm\frac{5}{2}\sqrt{3} + \frac{3}{2} \pm \sqrt{17 \pm \frac{21}{2}\sqrt{3}}.$$

7. Given $x^4 - 3x^2 - 4x - 3 = 0$, to find the values of x.

$$\textit{Anſ.}\ x = \frac{1 \pm \sqrt{3}}{2}\ \textit{and}\ \frac{-1 \pm \sqrt{-3}}{2}.$$

Here, by comparing this with the general equation $x^4 + ax^3 + bx + cx + d = 0$, we ſhall have $a = 0$, $b = 0$, $c = 12$, and $d = -17$; and therefore $k\,(= \frac{1}{4}ac - d) = 17$, $l\,(= \frac{1}{4}c^2 + d \times \overline{\tfrac{1}{4}a^2 - b}) = 36$, and $A^3 - \frac{1}{2}bA^2 + kA - \frac{1}{2}l = A^3 + 17A - 18 = 0$.

And, from this equation, A will be found equal to 1; and therefore $B\,(= \overline{2A + \tfrac{1}{4}a^2 - b}\Big|^{\frac{1}{2}}) = \sqrt{2}$, $C\left(= \frac{aA - c}{2B}\right) = \frac{-12}{2\sqrt{2}} = -3\sqrt{2}$, and $x = \pm\frac{1}{2}\sqrt{2} \pm \overline{\tfrac{1}{2} \mp 3\sqrt{2} - 1}\Big|^{\frac{1}{2}} = \pm\frac{1}{2}\sqrt{2} \mp \overline{\mp 3\sqrt{2} - \tfrac{1}{2}}\Big|^{\frac{1}{2}}$.

Hence, the four roots of the equation are $\frac{1}{2}\sqrt{2} + \overline{-3\sqrt{2} - \tfrac{1}{2}}\Big|^{\frac{1}{2}}$, $\frac{1}{2}\sqrt{2} - \overline{-3\sqrt{2} - \tfrac{1}{2}}\Big|^{\frac{1}{2}}$, $-\frac{1}{2}\sqrt{2} + \overline{3\sqrt{2} - \tfrac{1}{2}}\Big|^{\frac{1}{2}}$, and $-\frac{1}{2}\sqrt{2} - \overline{3\sqrt{2} - \tfrac{1}{2}}\Big|^{\frac{1}{2}}$; the firſt and ſecond of which are impoſſible.

In ſome particular caſes of this rule, the roots may be found by means of quadratics only.

Several other methods of ſolving biquadratic equations have been invented by different authors; but one of the moſt ingenious is that given by M. *Euler*, in page 664 of his *Elémens D' Algebre.*

Equations of five, or more dimenſions, may be reduced to thoſe of an inferior degree; but the proceſs will be exceedingly tedious, as no general rule can be given for reſolving them.

PROBLEM VII.

To find the roots of equations in general, by the methods of approximation and converging ſeries.

RULE.*

1. Find, by trial, a number nearly equal to the root required.

2. Call the number, thus found, r, and let z be put equal to the difference between r and the true root x.

3. Inſtead of x, in the given equation, ſubſtitute its equal $r \pm z$, and there will ariſe a new equation, affected only with z and known quantities.

4. Reject all thoſe quantities in which there are two or more dimenſions of z, and the value of z will be found by means of a ſimple equation.

5. Add the value of z, thus found, to the value of r, and it will give the root required *nearly*.

6. If this root is not ſufficiently near the truth, repeat the operation, by ſubſtituting it inſtead of r, in the equation exhibiting the value of z, and it will give a *ſecond correction* for the root required.

EXAMPLES:

1. Given $x^2 - 5x - 31 = 0$, to find the value of x by approximation.

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* The rules hitherto given, for finding the roots of equations, are either very troubleſome and laborious, or elſe confined to particular caſes; but this method, by converging ſeries, is univerſal, extending to all kinds of equations whatever; and, though not accurately true, gives the value ſought to any aſſigned degree of exactneſs.

The method of obtaining the roots of equations, by approximation, was firſt made uſe of by *Vieta*.

The root, found by trial, is nearly equal to 8;
Let, therefore, $8=r$, *and* $r+z=x$; *then*

$$\begin{aligned} x^2 &= r^2+2rz+z^2 \\ -5x &= -5r-5z \\ -31 &= -31 \end{aligned}$$

$$r^2+2rz-5r-5z-31=0;$$

or $z=\frac{31+5r-r^2}{2r-5}=\frac{31+40-64}{16-5}=\frac{7}{11}=.6$, *and* $x=$ 8.6 *nearly.*

And, again, if 8.6 *be ſubſtituted in the place of* r *in the laſt equation, we ſhall have*

$z=\frac{31+5r-r^2}{2r-5}=\frac{31+43-73.96}{17.2-5}=\frac{.04}{12.2}=.0032$
and $x=8.6+.0032=8.6032$ *nearly.*

And, if this value be again ſubſtituted for r, *it will give* $z=.0000077808$, *and* $x=8.603277808$; *and ſo on to any degree of exactneſs required.*

2. Given $x^3+x^2+x=90$, to find the value of x by approximation.

The root, found by trial, is nearly equal to 4;
Let, therefore, $4=r$, *and* $r+z=x$, *then,*

$$\begin{aligned} x^3 &= r^3+3r^2z+3rz^2+z^3 \\ x^2 &= r^2+2r\,z+z^2 \\ x &= r+z \end{aligned}$$

$$r^3+3r^2z+r^2+2rz+r+z=90;$$

or, $z=\frac{90-r^3-r^2-r}{3r^2+2r+1}=\frac{90-64-16-4}{48+8+1}=\frac{6}{57}=.10$,
and $x=4.1$ *nearly.*

And again, if 4.1 *be ſubſtituted in the place of* r *in the laſt equation, we ſhall have*

$\frac{90-r^3-r^2-r}{3r^2+2r+1}=\frac{90-68.921-16.81-4.1}{50.43+8.2+1}=.00283$, *and* $x=4.1+.00283=4.10283$ *nearly; and ſo on to any degree of exactneſs required.*

3. Given $x^2+20x=100$, to find the value of x by approximation. *Anſ.* $x=4.1421356236$.

4. Given $x^3+10x^2+5x=2600$, to find the value of x by approximation. *Anſ.* 10.1794653.

5. Given $x^3+2x^2-23x-70=0$, to find the value of x. *Anſ.* $x=5.1349$.

6. Given $x^3-15x^2+63x-50=0$, to find the value of x. *Anſ.* $x=1.02803923127$.

7. Given $x^4-3x^2-75x=10000$, to find the value of x. *Anſ.* $x=9.8860027$.

8. Given $x^5+2x^4+3x^3+4x^2+5x=54321$, to find the value of x. *Anſ.* $x=8.1414455$.

RULE II.

1. Aſſume the general equation $az+bz^2+cz^3+dz^4$, &c. $=p$; where z is the converging quantity, and a, b, c, d, &c. co-efficients whoſe values are known.

2. Then will $\frac{ap}{a^2+bp}$ be an approximation of the firſt degree.

3. And, if s be put $=\frac{b}{a}-\frac{c}{b}$ we ſhall have $\frac{\overline{a+sp}\times p}{\overline{a^2+b+as}\times p}$ for an approximation of the ſecond degree.

4. And, in like manner, if w be put $=\frac{2b}{a}+$

$\frac{ad-bc}{b^2-ac}$, then will $\frac{ap\times\overline{a+wp}}{a\times\overline{a^2+b+aw}\times p+\overline{w-s}\times p^2}$ be an approximation of the third degree, &c.

EXAMPLES:

1. Given $x^2+20x=100$, to find the value of x.

The root, found by trial, is nearly equal to 4;

Let therefore $4+z=x$, *and, by ſubſtitution, the equation will become* $28z+z^2=4$.

Whence, from the rule, $a=28$, $b=1$, $c=0$, *&c. and* $p=4$.

Therefore $z=\frac{ap}{a^2+bp}=\frac{112}{788}=\frac{28}{197}=.14213$ *for the firſt approximation.*

And, ſince $s=\frac{b}{a}-\frac{c}{b}=\frac{1}{28}$, *we ſhall have*

$$z=\frac{\overline{a+sp}\times p}{\overline{a^2+b+as}\times p}=\frac{\overline{28+\frac{1}{7}}\times 4}{\overline{28\times28+1+1}\times 4}=\frac{28+\frac{1}{7}}{28\times7+2}=$$

$\frac{197}{1386}=.14213564$ *for the ſecond approximation.*

And, in like manner, find $w=\frac{2b}{a}+\frac{ad-bc}{b^2-ac}=\frac{1}{14}$, *it will be* $z=\frac{ap\times\overline{a+wp}}{a\times\overline{a^2+b+aw}\times p+\overline{w-s}\times p^2}=$

$$\frac{28\times4\times\overline{28+\frac{2}{7}}}{28\times\overline{784+12+\frac{4}{7}}}=\frac{28\times\overline{28+\frac{2}{7}}}{7\times\overline{796+\frac{1}{7}}}=\frac{28\times198}{49\times796+1}=$$

$\frac{5544}{39005}=.1421356236$; *or* $x=4.1421356236$ *for the root required, extremely near.*

2. Given $x^2+4x-10=100$, to find the value of x. *Anſ.* 10.677078.

3. Given $x^3-17x^2+54x=350$, to find the value of x. *Anſ.* 14.95407.

4. Given $x^3-2x-5=0$, to find the value of x. *Anſ.* 2.09455147.

5. Given $2y^4-16y^3+40y^2-30y=-1$, to find the value of y. *Anſ.* $y=1.284724$.

PROBLEM VIII.

To extract the root of any pure power in numbers.

RULE.

1. Let $m=$ the number whoſe root is required; $r=$ neareſt root which can be found by trial; and $n=$ to the index.

2. Then, by putting $v=\frac{nr^n}{m-r^n}$, we ſhall have

$$x=r+\frac{r\times\overline{6v+n+1}}{v\times\overline{6v+4n-2}}=\text{root, } \textit{nearly}; \text{ or } x=r+\frac{r\times\overline{2v+n}}{v\times\overline{2v+2n-1}+\frac{1}{6}\times\overline{n-1}\times\overline{2n-1}} \textit{ extremely near.}$$

EXAMPLES:

1. Given $x^2=2$, or, which is the ſame thing, let the ſquare root of 2 be found.

Suppoſe the root, found by trial, to be 1.4; *then we ſhall have* $m=2$, $r=1.4$, $n=2$, *and* $v=\frac{2\times1.96}{2-1\,96}=98$.

And, therefore, $x=r+\frac{r\times\overline{6v+\;\overline{n+1}}}{v\times\overline{6v+4n-2}}=1.4+\frac{1.4\times591}{98\times594}=1.4+\frac{197}{70\times198}=1.4+\frac{197}{13860}=1.41421356=$ *root required nearly.*

And, if the ſecond approximation be uſed, the root will be found $=1.41421356236$, *which is true to the laſt place of decimals.*

2. Given $x^3=500$; or let it be required to extract the cube root of 500.

Suppoſe the root, found by trial, to be 8; *then we ſhall have* $m=500$, $r=8$, $n=3$, *and* $v=\frac{3\times512}{-12}=-128$;

And, therefore $x=r+\frac{r\times\overline{6v+\;\overline{n+1}}}{v\times\overline{6v+4n-2}}=8-.063=7.93$ *for the 1ſt approximation.*

Or $x=r+\frac{r\times\overline{2v+n}}{\overline{2v+2n-1}\times v+\frac{1}{6}\times\overline{n-1}\times\overline{2n-1}}=8-\frac{6072}{96389}=7.937005259936$ *for the 2d approximation, which is true to the laſt place of decimals.*

3. Let it be required to find the cube root of 2.
Anſ. 1.259921049894.

4. What is the cube root of 117? *Anſ* 4.89097.

5. What is the ſurſolid, or 5th root, of 125000?
Anſ. 10.456389.

6. It is required to find the 7th root of 100000.
Anſ. 5.1794746792.

PROBLEM IX.

To find the root of an exponential equation.

RULE.*

1. Find, by trial, two numbers, as near the true root as poſſible, and ſubſtitute them in the given equation inſtead of the unknown quantity, marking the errors which ariſe from each of them.

2. Multiply the difference of the two numbers, found by trial, by the leaſt error, and divide the product by the difference of the errors, when they are alike, and by their ſum when they are unlike.

3. Add the quotient, laſt found, to the number belonging to the leaſt error, when that number is too little, and ſubtract it when too great, and the reſult will give the true root *nearly*.

4. Take this root and the neareſt of the former, and, by proceeding in like manner, a root will be had ſtill nearer than before; and ſo on to any degree of exactneſs required.

EXAMPLES.

1. Given $x^x = 100$, to find the value of x by approximation.

By the nature of logarithms $x \times log.\ x = log.\ 100 = 2$.

And, ſince x *is found, by trial, to be greater than* 3 *and leſs than* 4;

Let, therefore, 3.5 *and* 3.6 *be the two ſuppoſed values of* x.

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[*] The rule for ſolving exponential equations was invented by M. *Jean Bernoulli*, and publiſhed in the *Leipſic* Acts, 1697.

Then the log. $x=$ *log.* $3.5=.5440680$; *and* $x\times$ *log.* $x=1.9042380$

2

$-.0957620=$ 1*ſt error, too little.*

And the log. $x=$ *log.* $3.6=.5563025$; *and* $x\times$ *log.* $x=2.0026890$

2

$.0026890=$ 2*d error, too great.*

1*ſt number* 3.5 — 1*ſt error* $-.095762$

2*d number* 3.6 — 2*d error* $+.002689$

$0.1=$ *diff.* — $.098451=$ *ſum.*

$\frac{0.1\times.002689}{.098451}=.00273=$ *correction,*

2*d number* 3.60000

correction $-.00273$

$3.59727=x=$ *root nearly.*

Again, ſuppoſe $x=3.597$; *then we ſhall have*

log. $x=\ .5559404$, *and*

$x\times$ *log.* $x=1.9997176$, *which ſubtracted from* 2

gives .0002824 *the* 3*d error, too little.*

2*d number* 3.600 — 2*d error* $+.0026890$

3*d number* 3.597 — 3*d error* $-.0002824$

$.003=$ *diff.* — $.0029714=$ *ſum.*

$\frac{.003\times.0002824}{.0029714}=.000285$ *the correction.*

3*d number* 3.597000

correction $+0.000285$

$3.597285=x=$ *root required nearly.*

OF INDETERMINATE, OR UNLIMITED PROBLEMS.

A problem is ſaid to be *indeterminate* or *unlimited*, when the equations, expreſſing the conditions of a queſtion, are leſs in number than the unknown quantities to be determined. And though ſuch kind of problems are capable of innumerable anſwers, yet the reſults, in whole numbers, are generally limited to ſome determinate number, and may be obtained as follows.

PROBLEM I.

To find the values of x *and* y*, in the equation* $ax = by + c$; *where* a, b, *and* c, *are given numbers, which admit of no common diviſor.*

RULE.

1. Let *wh.* ſtand for a *whole number*, and reduce the equation to $x = \frac{by+c}{a} = wh.$

2. Make $\frac{by+c}{a} =$ to $\frac{dy+f}{a}$, by throwing all whole numbers out of it, till d and f be each leſs than a.

3. Subtract $\frac{dy+f}{a}$, or ſome multiple of it, from $\frac{ay}{a}$, $\frac{2ay}{a}$, $\frac{3ay}{a}$, or any other multiple of y, that comes

near the former, and the remainder will be a whole number.

4. Take this remainder, or any multiple of it, from ſome of the foregoing fractions, or from any whole number, which is nearly equal to it, and the remainder, in this caſe, will alſo be a whole number.

5. Proceed in the ſame manner with this laſt remainder; and ſo on till the co-efficient of y becomes equal to 1; or $\frac{y+g}{a} = wh. = p$.

6. Then will $y = ap - g$; where p may be any whole number whatever; and y being known, x may be found from the given equation.

EXAMPLES:

1. Given $19x = 14y - 11$, to find x and y in whole numbers.

Firſt, $x = \frac{14y-11}{19} = wh.$; *and* $\frac{19y}{19} = wh.$

Then, by ſubtraction, $\frac{19y}{19} - \frac{14y-11}{12} = \frac{5y+11}{19} = wh.$

Again, $\frac{5y+11}{19} \times 4 = \frac{20y+44}{19} = \frac{20y+6}{19} + 2 = wh.$

or $\frac{20y+6}{19} = wh.$

Therefore $\frac{20y+6}{19} - \frac{19y}{19} = \frac{y+6}{19} = wh. = p.$

And $y = 19p - 6$; *where, if* p *be taken* $= 1$, *for the leaſt affirmative value of* y, *we ſhall have* $y = 13$, *and* $x = 9$, *the anſwer.*

2. Given $3x=8y-16$, to find the values of x and y in whole numbers.

Here, $x=\frac{8y-16}{3}=2y-5+\frac{2y-1}{3}=$ *wh. or* $\frac{2y-1}{3}$ $=$ *wh.*

And, $\frac{2y-1}{3}\times 2=\frac{4y-2}{3}=$ *wh. But* $\frac{3y}{3}$ *is alſo* $=$ *wh.*

Therefore $\frac{4y-2}{3}-\frac{3y}{3}=\frac{y-2}{3}=$ *wh.* $=p$.

Whence $y=3p+2$; *and, by taking* $p=0$, *we ſhall have* $y=2$ *and* $x=0$, *the anſwer.*

3. Given $9x+13y=2000$; to find all the poſſible values of x and y in whole numbers.

Firſt, $x=\frac{2000-13y}{9}=222-y+\frac{2-4y}{9}=$ *wh. or* $\frac{2-4y}{9}=$ *wh.*

And, $\frac{2-4y}{9}\times 2=\frac{4-8y}{9}=$ *wh. But,* $\frac{9y}{9}$ *is alſo* $=$ *wh.*

Therefore $\frac{9y}{9}+\frac{4-8y}{9}=\frac{y+4}{9}=$ *wh.* $=p$.

Whence $y=9p-4$; *and, by taking* $p=1$, *we ſhall have* $y=5$ *and* $x=215$.

And, by adding 9, *continually, to the laſt value of* y, *and ſubtracting* 13 *from that of* x, *all the poſſible anſwers will ſtand as follows:*

$$x=\left\{\begin{array}{l}215.189.163.137.111.85.59.33\\202.176.150.124.\ 98.72.46.20.7\end{array}\right.$$

$$y=\left\{\begin{array}{l}5.23.41.59.77.95.113.131\\14.32.50.68.86.104.122.140.149\end{array}\right.$$

4. Given $24x=13y+16$, to find x and y in whole numbers. *Anſ.* $x=5$, *and* $y=8$.

5. Given $14x=4y+7$, to find x and y in whole numbers. *Anſ. The queſtion is impoſſible.*

6. Given $27x=1600-16y$, to find x and y in whole numbers. *Anſ.* $y=19$, *and* $x=48$.

7. Given $87x+256y=15410$, to find the leaſt value of x, and the greateſt of y, in whole poſitive numbers. *Anſ.* $x=30$, *and* $y=12800$.

8. Given $5x+7y+11z=224$, to find all the poſſible values of x, y, and z, in whole numbers.

Anſ. The number of anſwers is 60.

9. How many different ways is it poſſible to pay 100l. in guineas and piſtoles only; a guinea being equal to 21s. and a piſtole to 17s.

Anſ. 6 *different ways.*

10. To determine whether it be poſſible to pay 100l. in guineas and moidores only.

Anſ. The queſtion is impoſſible.

11. A perſon bought 20 birds, of three different ſorts, for 1s. 8d. the firſt ſort at 4d. the ſecond at $\frac{1}{2}$ d. and the third at $\frac{1}{4}$ a-piece: how many had he of each?

Anſ. 3 *of the 1ſt ſort,* 15 *of the 2d, and* 2 *of the 3d.*

12. I owe my friend a ſhilling, and have nothing about me but guineas, and he has nothing but louis d'ors; the queſtion is, how muſt I acquit myſelf of the debt? the louis d'ors being valued at 17s. *Anſ. I muſt pay him* 13 *guineas, and he muſt give me* 16 *louis d'ors.*

13. To find in what year of Chriſt the cycle of the ſun was 8, the cycle of the moon 10, and the cycle of indiction 10. *Anſ. In the year* 1567.

14. Twenty perſons, conſiſting of men, women, and children, ſpend altogether 20s. the men ſpend 4s. a-piece; the women 6d. and the children 3d. how many were there of each ſort?

Anſ. 3 *men,* 15 *women, and* 2 *children.*

15. A vintner has wine at 2s. 1s. 10d. and 1s. 6d. *per* gallon: how much of each ſort muſt he take, ſo as to make a mixture of 30 gallons, to be ſold at 1s. 8d. *per* gallon?

Anſ. 16, 2, 12; 17, 4, 9; 18, 6, 6; *or* 19, 8, 3 *of each ſort.*

16. To determine how many ways it is poſſible to pay 1000l. in crowns, guineas, and moidores, only. *Anſ.* 70734 *different ways.*

PROBLEM II.

To find ſuch a whole number x, as being divided by the given numbers a, b, c, &c. ſhall leave the given remainders f, g, h, &c.

RULE.

1. Subtract each of the remainders from x, and divide the difference by a, and there will reſult $\frac{x-f}{a}$, $\frac{x-g}{a}$, $\frac{x-h}{a}$, &c. = whole numbers.

2. Call the value of x, in the firſt fraction, p, and ſubſtitute this quantity in the place of x in the ſecond fraction.

3. Find the leaſt value of p, in the ſecond fraction, by the laſt problem, and call it r.

4. Let the value of x be found in terms of r, and ſubſtitute this quantity in the place of x in the third fraction.

5. Find the leaſt value of r in the third fraction, and call it s; and the value of x in terms of s, being ſubſtituted for x in the fourth fraction, and ſo on, will give the whole number required.

EXAMPLES:

1. To find the leaſt whole number, which, by being divided by 17, ſhall leave a remainder of 7; but being divided by 26, the remainder ſhall be 13.

Let $x=$ *number required.*

Then $\frac{x-7}{17}$, *and* $\frac{x-13}{26}=$ *whole numbers.*

And, by putting $\frac{x-7}{17}=p$, *we ſhall have* $x=17p+7$;
Which value of x, *being ſubſtitued in the 2d fraction,*
gives $\frac{17p-6}{26}=$ *wh. But* $\frac{26p}{26}$ *is alſo* $=$ *wh.*

And, therefore, $\frac{26p}{26}-\frac{17p-6}{26}=\frac{9p+6}{26}=$ *wh.*

or $\frac{9p+6}{26}\times 3=\frac{27p+18}{26}=p+\frac{p+18}{26}=$ *wh. or* $\frac{p+18}{26}$ $=$ *wh.* $=r$.

Hence $p=26r-18$, *and, by taking* $r=1$, *we ſhall have* $p=8$.

And, conſequently $x=17\times 8+7=143$, *the number required.*

2. To find a number which being divided by 11, 19, and 29, the remaiders ſhall be 3, 5, and 10.

Let $x=$ *number required.*

Then $\frac{x-3}{11}$, $\frac{x-5}{19}$ *and* $\frac{x-10}{29}=$ *whole numbers.*

And, by putting $\frac{x-3}{11}=p$, *we ſhall have* $x=11p+3$;

Which value of x, *being ſubſtituted in the 2d fraction, gives* $\frac{11p-2}{19}=$ *wh. or* $\frac{11p-2}{19}\times 2=\frac{22p-4}{19}=p+\frac{3p-4}{19}=$ *wh.*

Alſo, $\frac{3p-4}{19}=$ *wh.* $\frac{3p-4}{19}\times 6=\frac{18p-24}{19}=\frac{18p-5}{19}-1=$ *wh. or* $\frac{18p-5}{19}=$ *wh.*

But $\frac{19p}{19}$ *is, likewiſe,* $=$ *wh. and, therefore,* $\frac{19p}{19}-\frac{18p-5}{19}=\frac{p+5}{19}=$ *wh.* $=r$.

Hence, $p=19r-5$, *and* $x=\overline{19r-5}\times 11+3=209r-52$.

And, by ſubſtituting this value of x *in the 3d fraction, we ſhall have* $\frac{209r-62}{29}=7r-2+\frac{6r-4}{29}=$ *wh. or* $\frac{6r-4}{29}=$ *wh.*

But, $\frac{6r-4}{29}\times 5=\frac{30r-20}{29}=r+\frac{r-20}{29}=$ *wh. or* $\frac{r-20}{29}=$ *wh.* $=s$.

Therefore, $r=29s+20$; *and, by putting* $s=0$, *we ſhall have* $r=20$.

And, conſequently, $x=209\times 20-52=4128=$ *number required.*

3. To find the leaſt whole number, which being divided by 19, ſhall leave a remainder of 7; but being divided by 28, the remainder ſhall be 13.

Anſ. 349.

4. To find a number, which being divided by 3, 5, 7, and 2; will leave the remainders 2, 4, 6, and 0, reſpectively. *Anſ.* 104.

5. To find the leaſt whole number, which, being divided by 16, 17, 18, 19, and 20, ſhall leave 6, 7, 8, 9, and 10 remainders. *Anſ.* 232550.

6. To find the leaſt whole number, which, being divided by the nine digits, reſpectively, ſhall leave no remainders. *Anſ.* 2520.

DIOPHANTINE PROBLEMS.

* *Diophantine problems* are thoſe which relate to the finding of ſquare and cube numbers, &c. and are ſuch as are generally capable of a great variety of anſwers. They are ſo called from their inventor *Diophantus* of *Alexandria* in *Egypt*, who flouriſhed in or about the third century, and is the firſt writer on Algebra we meet with amongſt the ancients.

Theſe queſtions are ſo exceedingly curious and abſtruſe, that nothing leſs than the moſt refined Algebra, applied with the utmoſt ſkill and judgment, could ever ſurmount the difficulties which attend them. And, in this way, no man has ever extended the limits of the analytic art further than *Diophantus*, or diſcovered greater penetration and judgment in the application of it.

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* That *Diophantus* was not the inventor of algebra, as has been generally imagined, is exceedingly obvious; for his method of applying it is ſuch, as could only have been uſed in a very advanced ſtate of the ſcience. He no where ſpeaks of the fundamental rules and principles, as an inventor certainly would have done, but treats of it as an art already ſufficiently known; and ſeems to intend, not ſo much to teach it, as to cultivate and improve it, by ſolving ſuch queſtions as, before this time, had been thought too difficult to be ſurmounted.

When we conſider his work with attention, we are at a loſs which to admire moſt, his wonderful ſagacity, and peculiar artifice, in forming ſuch poſitions as the nature of the problems required, or the more than ordinary ſubtilty of his reaſoning upon them.

Every particular queſtion puts us upon a new way of thinking, and furniſhes a freſh vein of analytical treaſure, which cannot but be very inſtructive to the mind, in conducting it through almoſt all difficulties of this kind, wherever they occur.

The following method of reſolving theſe queſtions will be found of conſiderable ſervice; but no general rule can be given, that will ſuit all caſes; and therefore the ſolution muſt often be left to the ſagacity and ſkill of the learner.

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* It is probable, therefore, that Algebra was known in the world, long before the time of *Diophantus*; but that the works of preceding writers have been deſtroyed by the ravages of time, or the depredations of ignorant barbarians.

His *Arithmetics*, out of which theſe problems were moſtly collected, conſiſted originally of thirteen books; but the firſt ſix only are now extant. The beſt edition is ſaid to be that publiſhed at *Paris*, by Monſieur *Bachet*, in the year 1621. And in this work, the ſubject is ſo ſkilfully handled, that the moderns, notwithſtanding their other improvements, have been able to do little more than explain and illuſtrate his method.

Thoſe who have ſucceeded beſt, in this reſpect, are *Vieta*, *Brancker*, *Kerſey*, *De Billy*, *Ozanam*, *Preſtet*, *Saunderſon*, *Fermat*, and *Euler*.

RULE.

1. For the root of the ſquare or cube required, put one or more letters ſuch, that when they are involved, either the given number, or the higheſt power of the unknown quantity, may vaniſh from the equation; and then if the unknown quantity be but of one dimenſion, the problem will be ſolved by reducing the equation.

2. But if the unknown quantity be ſtill a ſquare, or a higher power, ſome other new letters muſt be aſſumed to denote the root; with which proceed as before; and ſo on till the unknown quantity be but of one dimenſion; and from this all the reſt will be determined.

EXAMPLES:

1. [*] To divide a given ſquare number (100) into two ſuch parts, that each of them may be a ſquare number.

Let x^2 ($=\Box$) *be one of the parts, and then* $100-x^2$ *will be the other part, which is alſo to be a ſquare number.*

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[*] If $x-10$ had been made the ſide of the ſecond ſquare, in this queſtion, inſtead of $2x-10$, the equation would have been $x^2-20x+100=100-x^2$; in which caſe, x, the ſide of the firſt ſquare, would have been found $=10$, and $x-10$, or the ſide of the ſecond ſquare $=0$; and for this reaſon the ſubſtitution, $x-10$, was avoided; but $3x-10$, $4x-10$, or any other quantity of the ſame kind, would have ſucceeded equally as well as the former; though, in ſome caſes, the reſults would have been leſs ſimple.

Aſſume the ſide of this ſecond ſquare $=2x-10$.
Then will $100-x^2=\overline{2x-10}|^2=4x^2-40x+100$;
And, by reduction, $x=8$, *and* $2x-10=6$.
Therefore 64 *and* 36 *are the parts required.*

THE SAME GENERALLY.

Let $a^2=$ *given ſquare number,* $x^2 (=\square) =$ *one of its parts, and* $a^2-x^2=$ *the other, which is alſo to be a ſquare number.*

Aſſume the ſide of this ſecond ſquare $=rx-a$,
then will $a^2-x^2=\overline{rx-a}|^2=r^2x^2-2arx+a^2$;
And, by reduction, $x=\frac{2ar}{r^2+1}$, *and* $rx-a=\frac{2ar^2}{r^2+1}-a=\frac{ar^2-a}{r^2+1}$.

Therefore $\overline{\frac{2ar}{r^2+1}}\Big|^2$ *and* $\overline{\frac{ar^2-a}{r^2+1}}\Big|^2$ *are the parts required; where* a *and* r *may be any numbers, taken at pleaſure.*

* 2. To divide a given number (13) conſiſting of two known ſquare numbers (9 and 4) into two other ſquare numbers.

If s and r be any two unequal numbers, of which s is the greater; then will $2rs$, s^2-r^2 and s^2+r^2, be the perpendicular, baſe, and hypothenuſe of a right-angled triangle.

And from this canon two ſquare numbers may be found, whoſe ſum or difference ſhall be ſquare numbers; for $\overline{2rs}|^2+\overline{s^2-r^2}|^2=\overline{s^2+r^2}|^2$, and $\overline{r^2+s^2}|^2-\overline{2rs}|^2=\overline{s^2-r^2}|^2$, or $\overline{s^2+r^2}|^2-\overline{s^2-r^2}|^2=\overline{2rs}|^2$; and this when s and r are any numbers whatever.

* This queſtion is conſidered, by *Diophantus*, as a very important one, being made the foundation, or *baſis*, of moſt

And, conſequently, $y=\frac{r^2-2rs^2}{4rs+1}$ *and* $x=\frac{r^2-y}{2r}=\frac{2r^2s+s^2}{4rs+1}$.

So that $\frac{r^2-2rs^2}{4rs+1}$ *and* $\frac{2r^2s+s^2}{4rs+1}$ *are the numbers required; where r and s may be taken at pleaſure, provided that r be greater than* $2s^2$.

5. To find two numbers, whoſe ſum and difference ſhall be both ſquare numbers.

Let x *and* x^2-x *be the two numbers ſought.*

Then, ſince their ſum is evidently a ſquare number, one of the conditions of the queſtions will be anſwered.

There remains, therefore, only their difference x^2-2x *to be made a ſquare.*

And, if for the ſide of this ſquare, there be put $x-r$, *we ſhall have* $x^2-2rx+r^2=x^2-2x$, *or* $2rx-2x=r^2$.

Whence, $x=\frac{r^2}{2r-2}$ *and* $x^2-x+\overline{\frac{r^2}{2r-2}}\Big|^2-\frac{r^2}{2r-2}$.

So that $\frac{r^2}{2r-2}$ *and* $\overline{\frac{r^2}{2r-2}}\Big|^2-\frac{r^2}{2r-2}$ *are the numbers required; where r may be taken at pleaſure, provided it be greater than* 1.

6. To find three numbers ſuch, that not only the ſum of all three of them, but alſo the ſum of every two ſhall be a ſquare number.

Let $4x$, x^2-4x *and* $2x+1$ *be the three numbers ſought.*

Then $\overline{4x+x^2-4x}\,(x^2)$, $\overline{x^2-4x+2x+1}\,(x^2-2x+1)$, *and* $\overline{4x+x^2-4x+2x+1}\,(x^2+2x+1)$, *are evidently ſquares.*

And, therefore, three of the conditions, mentioned in the queſtion, are accompliſhed.

Whence, it remains only, to make the quantity $\overline{4x}+\overline{2x+1}$, *or* $6x+1=$ *to a ſquare.*

Let, therefore, $6x+1=a^2$; *and we ſhall have* $x=\frac{a^2-1}{6}$.

And, conſequently, $\frac{4a^2-4}{6}$, $\left.\overline{\frac{a^2-1}{6}}\right|^2-\frac{4a^2-4}{6}$, *and* $\frac{2a^2-2}{6}+1$; *or* $\frac{2a^2-2}{3}$, $\frac{a^4-26a^2+25}{36}$, *and* $\frac{a^2+2}{3}$ *are the numbers required; where a may be taken at pleaſure, provided it be greater than* 5.

7. To find three ſquare numbers ſuch, that the ſum of every two of them ſhall be a ſquare number *.

Let x^2, y^2, *and* z^2 *be the numbers ſought;*

Then $x^2+z^2=\square$, $y^2+z^2=\square$, *and* $x^2+y^2=\square$.

Or $\frac{x^2}{z^2}+1=\square$, $\frac{y^2}{z^2}+1=\square$, *and* $\frac{x^2}{z^2}+\frac{y^2}{z^2}=\square$.

And, by putting $\frac{x}{z}=\frac{s^2-1}{2s}$, *and* $\frac{y}{z}=\frac{r^2-1}{2r}$, *we ſhall have* $\frac{x^2}{z^2}+1=\frac{s^4+2s^2+1}{4s^2}$, *and* $\frac{y^2}{z^2}+1=\frac{r^4+2r^2+1}{4r^2}$ *which are both evidently ſquares; and*

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* This queſtion is capable of a great variety of anſwers; but the leaſt roots, which have yet been found, in whole numbers, are 44, 117, and 240. See *Elémens d'Algebre, par M. Euler,* tome II. page 327.

therefore it remains only to make $\frac{x^2}{z^2}+\frac{y^2}{z^2}=$ *ſquare number.*

But $\frac{x^2}{z^2}+\frac{y^2}{z^2}=\overline{\frac{s^2-1}{2s}}\Big|^2+\overline{\frac{r^2-1}{2r}}\Big|^2=\frac{\overline{s^2-1}|^2}{4s^2}+\frac{\overline{r^2-1}|^2}{4r^2}$
$=\frac{4r^2\times\overline{s^2-1}|^2+4s^2\times\overline{r^2-1}|^2}{4r^2s^2}=\square$;

Or $r^2\times\overline{s^2-1}|^2+s^2\times\overline{r^2-1}|^2=r^2\times\overline{s+1}|^2\times\overline{s-1}|^2+s^2\times\overline{r+1}|^2\times\overline{r-1}|^2=\square$.

And, by making $r-1=s+1$, *or* $r=s+2$, *we ſhall have* $\overline{s+2}|^2\times\overline{s+1}|^2\times\overline{s-1}|^2+s^2\times\overline{s+3}|^2\times\overline{s+1}|^2=\square$;

Or $\overline{s+2}|^2\times\overline{s-1}|^2+s^2\times\overline{s+3}|^2=2s^4+8s^3+6s^2-4s+4=\square$.

Now, let the root of this ſquare be aſſumed $=\frac{5}{4}s^2-s+2$,

Then, $2s^4+8s^3+6s^2-4s+4=\overline{\frac{5}{4}s^2-s+2}|^2=\frac{25}{16}s^4-\frac{5}{2}s^3+5s^2+s^2-4s+4$; *or* $2s^4+8s^3=\frac{25}{16}s^4-\frac{5}{2}s^3$; *or* $2s+8=\frac{25}{16}s-\frac{5}{2}$.

Whence $s=-24$, *and* $r=-22$.

And, conſequently, $\frac{x}{z}=\frac{s^2-1}{2s}=-\frac{575}{48}$, *and* $\frac{y}{z}=\frac{r^2-1}{2r}=-\frac{483}{44}$;

or $x=-\frac{575z}{48}$, *and* $y=-\frac{483z}{44}$.

In order, therefore, to have the anſwer in whole numbers, let $z=528$, *and we ſhall have* $x=6325$, *and* $y=5796$, *or* 528, 5796 *and* 6325 *for the roots of the ſquares required.*

8. To find a number x ſuch, that $x+1$ and $x-1$ ſhall be both ſquare numbers. *Anſ.* $x=\frac{5}{4}$.

9. To find a number x ſuch, that $x+128$ and $x+192$ ſhall be both ſquares. *Anſ.* $x=97$.

10. To find a number x ſuch, that x^2+x and x^2-x may be both ſquares. *Anſ.* $x=\frac{25}{24}$.

11. To find two numbers x and y ſuch, that $x+y$, x^2+y *and* y^2+x may be all ſquares.
Anſ. $x=\frac{1}{6}$ *and* $y=\frac{1}{12}$.

12. To find three numbers in arithmetical progreſſion ſuch, that the ſum of every two of them may be a ſquare number. *Anſ.* $120\frac{1}{2}$, $840\frac{1}{2}$, *and* $1560\frac{1}{2}$.

13. To find three numbers ſuch, that if to the ſquare of every one of them the ſum of the other two be added, the three ſums ſhall be all ſquares.
Anſ. $\frac{8}{3}$, $\frac{16}{3}$ *and* 1.

14. To find two numbers in proportion as 8 is to 15, and ſuch that the ſum of their ſquares ſhall make a ſquare number. *Anſ.* 576 *and* 1080.

15. To find three ſquare numbers in arithmetical progreſſion. *Anſ.* 1, 25, *and* 49.

16. To find three ſquare numbers in harmonical proportion. *Anſ.* 1225, 49, *and* 25.

17. To find four numbers ſuch, that if a ſquare number (100) be added to the product of every two of them, the ſums ſhall be all ſquares.
Anſ. 2, 32, 88 *and* 168.

18. To find two numbers ſuch, that their difference may be equal to the difference of their ſquares, and that the ſum of their ſquares ſhall be a ſquare number. *Anſ.* $\frac{4}{7}$ *and* $\frac{3}{7}$.

19. To find three numbers in geometrical proportion ſuch, that every one of them being increaſed by a given number 19, ſhall make ſquare numbers.
Anſ. 81, $\frac{5}{4}$ *and* $\frac{25}{1296}$.

20. To find two numbers ſuch, that if their product be added to the ſum of their ſquares, it ſhall make a ſquare number.
Anſ. 5 *and* 3, 8 *and* 7, 16 *and* 5, &c.

21. To divide a given number (10) into four ſuch parts, that the ſum of every three of them may make a ſquare number. *Anſ.* 1, 6, $\frac{186}{289}$ *and* $\frac{681}{289}$.

22. To find three ſquare numbers ſuch, that their ſum being ſeverally added to their three ſides ſhall make ſquare numbers.

Anſ. $\frac{4418}{62920}$, $\frac{13254}{62920}$, *and* $\frac{19881}{62920}$ = *roots required.*

23. To find two numbers ſuch, that their ſum being increaſed and leſſened, either by their difference, or the difference of their ſquares, the ſums and remainders ſhall be all ſquares.

Anſ. $\frac{49}{50}$ *and* $\frac{1}{50}$.

24. To find two numbers ſuch, that not only each number, but alſo their ſum and their difference, being increaſed by unity, ſhall be all ſquare numbers. *Anſ.* 3024 *and* 5624.

25. To find three numbers ſuch, that whether their ſum be added to, or ſubtracted from the ſquare of every particular number, the numbers thence ariſing ſhall be all ſquares.

Anſ. $\frac{406}{96}$, $\frac{518}{96}$ *and* $\frac{791}{96}$.

26. To find three ſquare numbers ſuch, that the ſum of their ſquares ſhall alſo be a ſquare number.

Anſ. 9, 16, *and* $\frac{144}{25}$.

27. To find three ſquare numbers ſuch, that the difference of every two of them ſhall be a ſquare number. *Anſ.* 485809, 34225 *and* 23409.

28. To divide any given cube number (8) into three other cube numbers. *Anſ.* $\frac{64}{27}$, $\frac{125}{27}$ *and* 1.

29. Two cube numbers (8 and 1) being given, to find two other cube numbers, whoſe difference ſhall be equal to the ſum of the given cubes.

Anſ. $\frac{8000}{343}$, *and* $\frac{4913}{343}$.

30. To divide a given number (28) compoſed of two cube numbers (27 and 1) into two other cube numbers. *Anſ.* $\frac{63284705}{21446828}$ *and* $\frac{28340511}{21446828}$ *the roots.*

31. To find three cube numbers ſuch, that if from every one of them a given number (1) be ſubtracted, the ſum of the remainders ſhall be a ſquare.

Anſ. $\frac{4913}{3375}$, $\frac{21952}{3375}$ *and* 8.

32. To find three numbers ſuch, that if they be ſeverally added to the cube of their ſum, the three ſums thence ariſing ſhall be all cubes.

Anſ. $\frac{1538}{157464}$, $\frac{18577}{157464}$ *and* $\frac{23625}{157464}$.

33. To find three numbers in arithmetical proportion ſuch, that the ſum of their cubes ſhall be a cube. *Anſ.* 3, 4, 5, *or* 149, 256, *and* 363, *&c.*

34. To find three cube numbers ſuch, that their ſum ſhall be a cube number.

Anſ. 3^3, 4^3, *and* 5^3, *or* 21^3, 19^3, 18^3, *&c.*

Of the SUMMATION, and INTERPOLATION, of INFINITE CONVERGING SERIES.

The *doctrine of infinite ſeries* is a ſubject which has engaged the attention of the greateſt Mathematicians in all ages; and is, perhaps, one of the moſt abſtruſe and difficult branches of abſtract mathematics.

To find the ſum of a ſeries, the number of whoſe terms is inexhauſtible, or infinite, has been conſidered by ſome as an extravagant paradox, or a thing impoſſible to be done. But this difficulty will be eaſily removed, by conſidering, that every finite magnitude whatever is diviſible *in infinitum*, or conſiſts of an infinite number of parts, whoſe aggregate, or ſum, is equal to the quantity firſt propoſed.

A number actually infinite is, indeed, a plain contradiction to all our ideas; for any number which we can poſſibly conceive, or of which we have any notion, muſt always be determinate and finite; ſo that a greater may be ſtill aſſigned, and a greater after this; and ſo on, without a poſſibility of ever coming to an end of the increaſe or addition.

And this inexhauſtibility, in the nature of numbers, is, therefore, all that we can diſtinctly comprehend by their infinity; for though we can eaſily conceive that a finite quantity may become greater and greater without end, yet we are not from thence enabled to form any notion of the *ultimatum*, or laſt magnitude, which is incapable of further augmentation.

We cannot, therefore, apply to an infinite ſeries the common notion of a ſum, or a collection of ſeveral particular numbers, that are joined and added together, one after another; for this ſuppoſes, that thoſe particulars are all known and determined. But as every ſeries generally obſerves ſome particular law, and continually approaches towards a term or limit, we can eaſily conceive it to be a whole, of its own kind, and that it muſt have a certain real value, whether that value be determinable or not.

Thus, in many ſeries, a number is aſſignable, beyond which, no number of its terms can ever reach, or indeed ever be equal to it; but yet may approach to it, in ſuch a manner, as to want leſs than any given difference. And this we may call the value, or ſum of the ſeries; not as being a number found by the common method of addition, but as ſuch a limitation of the value of the ſeries, taken in all its infinite capacity, that if it were poſſible to add them all together, one after another, the ſum would be equal to that number.

Again, in other ſeries, the value has no limitation; and this may be expreſſed by ſaying, that the ſum of the ſeries is infinitely great; or, which is the ſame thing, that it has no determinate and aſſignable value; but may be carried on, to ſuch a length, as that its ſum ſhall exceed any given number whatever.

According to the common rule for ſumming up a finite progreſſion of a geometric decreaſing ſeries, where r is the ratio, l the firſt term, and a the leaſt, the ſum is $\frac{rl-a}{r-1}$. And if we ſuppoſe a, the leſs extreme, to be actually decreaſed to 0, then the ſum of the whole ſeries will be $\frac{rl}{r-1}$. For it is demonſtrable, that the ſum of no aſſignable number of terms of the ſeries can ever be equal to that quotient; and yet no number leſs than it, will ever be equal to the value of the ſeries.

Whatever conſequences, therefore, follow from the ſuppoſition of $\frac{rl}{r-1}$ being the true and adequate value of the ſeries, taken in all its infinite capacity, as if all the parts were actually determined and added together, they can never be the occaſion of any aſſignable error, in any operation or demonſtration, where it is uſed in that ſenſe; becauſe if you ſay that it exceeds that value, it is demonſtrable that this exceſs muſt be leſs than any aſſignable difference, which is, in effect, no difference at all; and therefore, the ſuppoſed error will likewiſe be no error, and conſequently $\frac{rl}{r-1}$ may be looked upon as expreſſing the adequate and juſt value of the infinite ſeries.

But we are further ſatisfied of the reaſonableneſs of this doctrine, by finding, in fact, that a finite quantity does actually convert into an infinite ſeries, as appears in the caſe of circulating decimals. Thus, $\frac{2}{3}$ turned into a decimal is $=.6666$, &c. $=\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}+\frac{6}{10000}$, &c. continued *ad infinitum*. But this is plainly a geometric ſeries, from $\frac{6}{10}$, in the continued ratio of 10 to 1, and the ſum of all its terms, continued to infinity, will evidently be equal to $\frac{2}{3}$, or the number from whence it was originally derived. And the ſame may be ſhewn of many other ſeries, and of all circulating decimals in general.

PROBLEM I.

Any ſeries being given to find the ſeveral orders of differences.

RULE.

1. Take the firſt term from the ſecond, the ſecond from the third, the third from the fourth, &c. and the remainders will form a new ſeries, called the *firſt order of differences.*

2. Take the firſt term of this laſt ſeries from the ſecond, the ſecond from the third, the third from

the fourth, &c. and the remainders will form another new ſeries, called the *ſecond order of differences.*

3. Proceed, in like manner, for the *third*, *fourth*, *fifth*, &c. orders of differences; and ſo on till they terminate, or are carried as far as is thought neceſſary,

EXAMPLES:

1. To find the ſeveral orders of differences in the ſeries 1, 4, 9, 16, 25, 36, &c.

1, 4, 9, 16, 25, 36, &c.
3, 5, 7, 9, 11, &c. *1ſt diff.*
2, 2, 2, 2, &c. *2d diff.*
0, 0, 0, &c.

2. To find the ſeveral orders of differences in the ſeries 1, 8, 27, 64, 125, 216, &c.

1, 8, 27, 64, 125, 216, &c.
7, 19, 37, 61, 91, &c. *1ſt diff.*
12, 18, 24, 30, &c. *2d diff.*
6, 6, 6, &c. *3d diff.*
0, 0, &c.

3. To find the ſeveral orders of differences in the ſeries 1, 3, 6, 10, 15, 21, &c.

Anſ. *1ſt* 2, 3, 4, 5, *&c.* *2d* 1, 1, 1, *&c.*

4. To find the ſeveral orders of differences in the ſeries 1, 6, 20, 50, 105, 196, &c.

Anſ. *1ſt* 5, 14, 30, 55, 91, *&c.* *2d* 9, 16, 25, 36, *&c.* *3d* 7, 9, 11, *&c.* *4th* 2, 2, *&c.*

PROBLEM II.

Any ſeries a, b, c, d, e, &c. being given, to find the firſt term of the nth. order of differences.

RULE.

Let δ ſtand for the firſt term of the nth. differences.

Then will $a-nb+n\times\frac{n-1}{2}c-n\times\frac{n-1}{2}\times\frac{n-2}{3}d+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}e$, &c. to $n+1$ terms $=\delta$, when n is an even number.

And $-a+nb-n\times\frac{n-1}{2}c+n\times\frac{n-1}{2}\times\frac{n-2}{3}d-n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}e$, &c. to $n+1$ terms $=\delta$, when n is an odd number.

EXAMPLES:

1. Required the firſt term of the third order of differences, of the ſeries 1, 5, 15, 35, 70, &c.

Let $a, b, c, d, e,$ *&c.* $=1, 5, 15, 35, 70,$ *&c. and* $n=3$.

Then $-a+nb-n\times\frac{n-1}{2}c+n\times\frac{n-1}{2}\times\frac{n-2}{3}d=-a+3b-3c+d=-1+15-45+35=4=$ *the firſt term required.*

2. Required the firſt term of the fourth order of differences of the ſeries, 1, 8, 27, 64, 125, &c.

Let $a, b, c, d, e,$ *&c.* $=1, 8, 27, 64, 125,$ *&c. and* $n=4$.

Then $a-nb+n\times\frac{n-1}{2}c-n\times\frac{n-1}{2}\times\frac{n-2}{3}d+n\times$

$\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}e, =a-4b+6c-4d+e=1-32+162-256+125=0$; *ſo that the firſt term of the fourth order is* 0.

3. Required the firſt term of the fifth order of differences, of the ſeries 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c.
Anſ. $-\frac{1}{32}$.

4. Required the firſt term of the 8th order of differences of the ſeries, 1, 3, 9, 27, 81, &c. *Anſ.* 256.

PROBLEM III.

To find the nth. term of the ſeries; a, b, c, d, e, &c.

RULE.

Let d^{I}, d^{II}, d^{III}, d^{IV}, &c. be the firſt of the ſeveral orders of differences, found as in the laſt problem:

Then will $a+\frac{n-1}{1}d^{\mathrm{I}}+\frac{n-1}{1}\times\frac{n-2}{2}d^{\mathrm{II}}+\frac{n-1}{1}\times\frac{n-2}{2}\times\frac{n-3}{3}d^{\mathrm{III}}+\frac{n-1}{1}\times\frac{n-2}{2}\times\frac{n-3}{3}\times\frac{n-4}{4}d^{\mathrm{IV}}$, &c. be = nth. term required.

EXAMPLES.

1. To find the 12th term of the ſeries 2, 6, 12, 20, 30, &c.

2, 6, 12, 20, 30, &c.
4, 6, 8, 10, &c.
2, 2, 2, &c.
0, 0, &c.

Here 4 *and* 2 *are the firſt terms of the differences;*
Let, therefore, $4=d^{\text{I}}$, $2=d^{\text{II}}$, *and* $n=12$.

Then $a+\frac{n-1}{1}d^{\text{I}}+\frac{n-1}{1}\times\frac{n-2}{2}d^{\text{II}}=2+11d^{\text{I}}+55d^{\text{II}}=2+44+110=156=12$*th term required.*

2. Required the 20th term of the ſeries 1, 3, 6, 10, 15, 21, &c.

1, 3, 6, 10, 15, 21, &c.
2, 3, 4, 5, 6, &c.
1, 1, 1, 1, &c.
0, 0, 0, &c.

Here 2 *and* 1 *are the firſt terms of the differences.*
Let, therefore, $2=d^{\text{I}}$, $1=d^{\text{II}}$, *and* $n=20$,

Then $a+\frac{n-1}{1}d^{\text{I}}+\frac{n-1}{1}\times\frac{n-2}{2}d^{\text{II}}=1+19d^{\text{I}}+171d^{\text{II}}=1+38+171=210=20$*th term required.*

3. Required the 15th term of the ſeries 1, 4, 9, 16, 25, 36, &c. *Anſ.* 225.

4. Required the 20th term of the ſeries 1, 8, 27, 64, 125, &c. *Anſ.* 8000.

PROBLEM IV.

To find the ſum of n terms of the ſeries a, b, c, d, e, &c.

RULE.

Let d^{I}, d^{II}, d^{III}, d^{IV}, &c. be the firſt of the ſeveral orders of differences.

Then will $na+n\times\frac{n-1}{2}d^{\text{I}}+n\times\frac{n-1}{2}\times\frac{n-2}{3}d^{\text{II}}+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}d^{\text{III}}+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times$

$\frac{n-3}{4}\times\frac{n-4}{5}d^{\text{IV}}$, &c. $=$ to the ſum of n terms of the ſeries.

EXAMPLES:

1. To find the ſum of n terms of the ſeries 1, 2, 3, 4, 5, 6, &c.

1, 2, 3, 4, 5, 6, &c.
1, 1, 1, 1, 1, &c.
0, 0, 0, 0, &c.

Here 1 *and* 0 *are the firſt terms of the differences.*

Let, therefore, $a=1$, $d^{\text{I}}=1$, *and* $d^{\text{II}}=0$;

Then will $na+n\times\frac{n-1}{2}d^{\text{I}}=n+\frac{n^2-n}{2}=\frac{n^2+n}{2}=$ *ſum of* n *terms, as required.*

2. To find the ſum of n terms of the ſeries 1^2, 2^2, 3^2, 4^2, 5^2, &c. or 1, 4, 9, 16, 25, &c.

1, 4, 9, 16, 25, &c.
3, 5, 7, 9, &c.
2, 2, 2, &c.
0, 0, &c.

Here 3 *and* 2 *are the firſt terms of the differences:*

Let, therefore, $a=1$, $d^{\text{I}}=3$ *and* $d^{\text{II}}=2$.

Then will $na+n\times\frac{n-1}{2}d^{\text{I}}+n\times\frac{n-1}{2}\times\frac{n-2}{3}d^{\text{II}}=n+3n\times\frac{n-1}{2}+n\times\frac{n-1}{2}\times\frac{n-2}{3}=\frac{3n^2-n}{2}+\frac{n^3-3n^2+2n}{3}=\frac{n\times\overline{n+1}\times\overline{2n+1}}{6}=$ *ſum of* n *terms, as required.*

* 3. To find the ſum of n terms of the ſeries 1^3, 2^3, 3^3, 4^3, 5^3, &c. or 1, 8, 27, 64, 125, &c.

1, 8, 27, 64, 125, &c.
7, 19, 37, 61, &c.
12, 18, 24, &c.
6, 6, &c.
0, &c.

Here the firſt terms of the differences are 7, 12 and 6. Let, therefore, $a=1$, $d^{I}=7$, $d^{II}=12$, *and* $d^{III}=6$.

Then will $na+n\times\frac{n-1}{2}d^{I}+n\times\frac{n-1}{2}\times\frac{n-2}{3}d^{II}+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}d^{III}=n+7n\times\frac{n-1}{2}+12n\times\frac{n-1}{2}\times\frac{n-2}{3}+6n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}=\frac{7n^2-5n}{2}$

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* The ſums of a ſeries of powers of the natural numbers 1, 2, 3, 4, 5, &c. may be exhibited as follows:

1 $1+2+3+4$, &c. $+n=\frac{n^2+n}{2}$

2 $1^2+2^2+3^2+4^2$, &c. $+n^2=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$

3 $1^3+2^3+3^3+4^3$, &c. $+n^3=\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$

4 $1^4+2^4+3^4+4^4$, &c. $+n^4=\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$

5 $1^5+2^5+3^5+4^5$, &c. $+n^5=\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}$

6 $1^6+2^6+3^6+4^6$, &c. $+n^6=\frac{n^7}{7}+\frac{n^6}{2}+\frac{n^5}{2}-\frac{n^3}{6}+\frac{n}{42}$

7 $1^r+2^r+3^r+4^r$, &c. $+n^r=\frac{n^{r+1}}{r+1}+\frac{n^r}{2}+\frac{rn^{r-1}}{\text{[illegible]}4}-\frac{r.\overline{r-1}.\overline{r-2}}{2.3.4.5.6}n^{r-3}+\frac{r.\overline{r-1}.\overline{r-2}.\overline{r-3}.\overline{r-4}}{2.3.4.5.6.7.6}n^{r-5}-$&c.

$$+2n^3-6n^2+4n+\frac{n^4-6n^3+11n^2-6n}{4}=\frac{n^4+2n^3+n^2}{4}$$

= *ſum of n terms, as required.*

4. To find the ſum of n terms of the ſeries 2, 6, 12, 20, 30, &c. *Anſ.* $\frac{n\times\overline{n+1}\times\overline{n+2}}{3}$.

5. To find the ſum of n terms of the ſeries 1, 3, 6, 10, 15, &c. *Anſ.* $\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$.

6. To find the ſum of n terms of the ſeries 1, 4, 10, 20, 35, &c. *Anſ.* $\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}$.

7. To find the ſum of n terms of the ſeries 1^4, 2^4, 3^4, 4^4, &c. or 1, 16, 81, 256, &c.

Anſ. $\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$

The ſums of n terms of a ſeries of triangular numbers may alſo be exhibited as follows:

1	1+1+ 1+ 1, &c.	$=n$
2	1+2+ 3+ 4, &c.	$=\frac{n.\overline{n+1}}{1.2}$
3	1+3+ 6+10, &c.	$=\frac{n.\overline{n+1}.\overline{n+2}}{1.2.3}$
4	1+4+10+20, &c.	$=\frac{n.\overline{n+1}.\overline{n+2}.\overline{n+3}}{1.2.3.4}$
5	1+5+15+35, &c.	$=\frac{n.\overline{n+1}.\overline{n+2}.\overline{n+3}.\overline{n+4}}{1.2.3.4.5}$
6	1+6+21+56, &c.	$=\frac{n.\overline{n+1}.\overline{n+2}.\overline{n+3}.\overline{n+4}.\overline{n+5}}{1.2.3.4.5.6}$
	&c.	&c.

PROBLEM V.

The ſeries a, b, c, d, e, &c. being given, whoſe terms are at an unit's diſtance from each other; to find any intermediate term by interpolation.

RULE.

Let x be the diſtance of any term y to be interpolated, and d^{I}, d^{II}, d^{III}, &c. the firſt terms of the differences:

Then will $a+xd^{\mathrm{I}}+x\times\frac{x-1}{2}d^{\mathrm{II}}+x\times\frac{x-1}{2}\times\frac{x-2}{3}d^{\mathrm{III}}+x\times\frac{x-1}{2}\times\frac{x-2}{3}\times\frac{x-3}{4}d^{\mathrm{IV}}$ &c. $=y$.

EXAMPLES:

1. Given the logarithmic ſines of 1° 0′, 1° 1′, 1° 2′, and 1° 3′ to find the ſine of 1° 1′ 40″.

	1° 0′	1° 1′	1° 2′	1° 3′
Sines	8.2418553	8.2490332	8.2560943	8.2630424
		71779	70611	69481
			−1168	−1130
				38

Here the firſt terms of the differences are 71779, −1168, *and* 38.

Let, therefore, $x=1°\ 1'\ 40''-1°\ 0'=1'\ 40''=1\frac{2}{3}$ = *diſtance of* y, *the term to be interpolated*; *and* $d^{\mathrm{I}}=71779$, $d^{\mathrm{II}}=-1168$, *and* $d^{\mathrm{III}}=38$.

Then will $y=a+xd^{\mathrm{I}}+x\times\frac{x-1}{2}d^{\mathrm{II}}+x\times\frac{x-1}{2}\times\frac{x-2}{3}d^{\mathrm{III}}=a+\frac{5}{3}d^{\mathrm{I}}+\frac{5}{9}d^{\mathrm{II}}-\frac{5}{81}d^{\mathrm{III}}=8.2418553+$

$.0119631-.0000694-.0000002=8.2537533=$ *ſine of* $1^\circ\ 1'\ 40''$, *as was required.*

2. Given the ſeries $\frac{1}{50}, \frac{1}{51}, \frac{1}{52}, \frac{1}{53}, \frac{1}{54}$, &c. to find the term which ſtands in the middle between $\frac{1}{52}$ and $\frac{1}{53}$. *Anſ.* $\frac{1}{52.5}$.

3. Given the natural tangents of 88°, 54′, 88°, 55′, 88°, 56′, 88°, 57′, 88°, 58′, and 88°, 59′; to find the tangent of 88°, 58′, 18″. *Anſ.* 55.711144.

PROBLEM VI.

Having given a ſeries of equidiſtant terms a, b, c, d, e, &c. whoſe firſt differences are ſmall; to find any intermediate term by interpolation.

RULE.

Find the value of the unknown quantity in the equation which ſtands againſt the given number of terms, in the following table, and it will give the term required.

1	$a-b=0$
2	$a-2b+c=0$
3	$a-3b+3c-d=0$
4	$a-4b+6c-4d+e=0$
5	$a-5b+10c-10d+5e-f=0$
6	$a-6b+15c-20d+15e-6f+g=0$
7	$a-nb+n\times\frac{n-1}{2}c-n\times\frac{n-1}{2}\times\frac{n-2}{3}d$ &c. $=0$.

EXAMPLES:

1. Given the logarithms of 101, 102, 104, and 105; to find the logarithm of 103.

Here the number of terms are 4,
Therefore againſt 4, *in the table, we have* $a-4b+6c-4d+e=0$; *or* $c=\frac{4\times\overline{b+d}-\overline{a+e}}{6}$.

$$\text{Whence}\begin{cases} a= 2.0043214 \\ b= 2.0086002 \\ d= 2.0170333 \\ e= 2.0211893 \end{cases}$$

$$4\times\overline{b+d}=16.1025340$$
$$a+e= 4.0255107$$

$$6)12.0770233$$

$$2.0128372=\textit{log. of } 103, \textit{ as required.}$$

2. Given the cube roots of 45, 46, 47, 48, and 49; to find the cube root of 50. *Anſ.* 3.684033.

3 Given the logarithms of 50, 51, 52, 54, 55, and 56; to find the logarithm of 53.
Anſ. 1.7242758695.

PROMISCUOUS EXAMPLES RELATING TO SERIES.

1. To find the ſum (S) of n terms of the ſeries 1, 2, 3, 4, 5, 6, &c.

Firſt $1+2+3+4+5$, &c. .... $n=S$.
And $n+\overline{n-1}+\overline{n-2}+\overline{n-3}+\overline{n-4}$, &c. ....$1=S$.
Therefore $n+1+n+1+n+1+n+1$, &c. .... $\overline{n+1}=2S$.

And conſequently $\overline{n+1}\times n=2S$; *or* $S=\frac{n^2+n}{2}=$ *ſum required.*

To find the ſum (S) of n terms of the ſeries 1, 3, 5, 7, 9, 11, &c.

Firſt $1+3+5+7+9$, *&c.* .... $2n-1=S$.

And $\overline{2n-1}+\overline{2n-3}+\overline{2n-5}+\overline{2n-7}+\overline{2n-9}$, *&c.* .... $1=S$.

Therefore $2n+2n+2n+2n+2n$, *&c.*..... $2n=2S$.

And, conſequently, $2n\times n=2S$; *or* $S=\frac{2n\times n}{2}=n^2=$ *ſum required.*

3. Required the ſum (S) of n terms of the ſeries $a+\overline{a+d}+\overline{a+2d}+\overline{a+3d}+\overline{a+4d}$, *&c.*

Firſt $a+\overline{a+d}+\overline{a+2d}+\overline{a+3d}$, *&c.* .... $a+\overline{n-1}\times d=S$.

And $a+\overline{nd-d}+a+\overline{nd-2d}+a+\overline{nd-3d}+a+\overline{nd-4d}$, *&c.* .... $a=S$.

Therefore $2a+\overline{nd-d}+2a+\overline{nd-d}+2a+\overline{nd-d}$, *&c.* $2a+\overline{nd-d}=2S$.

And, conſequently, $\overline{2a+nd-d}\times n=2S$; *or* $S=\overline{2a+nd-d}\times\frac{n}{2}=$ *ſum required.*

OR THUS:

Firſt, $a+\overline{a+d}+\overline{a+2d}+\overline{a+3d}+\overline{a+4d}$, *&c.*

$$=\left\{\begin{array}{l}\overline{1+1+1+1+1,\ \&c.}\times a\\ \overline{+0+1+2+3+4,\ \&c.}\times d\end{array}\right\}=S.$$

But n terms of $1+1+1+1+1$, *&c.* $=n$.

Ditto of $0+1+2+3+4$, *&c.* $=\frac{n\times\overline{n-1}}{2}$

And therefore $S=na+\frac{n\times\overline{n-1}\times d}{2}=\overline{2a+nd-d}\times\frac{n}{2}$ *as before.*

4. To find the ſum (S) of n terms of the ſeries 1, x, x^2, x^3, x^4, &c.

Firſt, $1+x+x^2+x^3+x^4$, *&c.* . . . $x^{n-1}=S$.
And $x+x^2+x^3+x^4+x^5$, *&c.* . . . $x^n=Sx$.
Therefore $-1+x^n=Sx-S$;
Or $S=\frac{x^n-1}{x-1}=$ *ſum required.*

And, when x is a proper fraction, the ſum of the ſeries, continued ad infinitum, may be found in the ſame manner.

Thus, $1+x+x^2+x^3+x^4$, *&c.* $=S$.
And $x+x^2+x^3+x^4+x^5$, *&c.* $=Sx$.
Therefore $-1=Sx-S$; *or* $S-Sx=1$.

Whence $S=\frac{1}{1-x}=$ *ſum of an infinite number of terms, as required.*

5. Required the ſum (S) of the circulating decimal .999999, &c. continued *ad infinitum*.

Firſt, .999999, *&c.* $=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\frac{9}{10000}$ *&c.*
$=9\times:\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}$, *&c.* $=S$.
Or, $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}$, *&c.* $=\frac{S}{9}$.
Therefore, $1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}$, *&c.* $=\frac{10S}{9}$.
And, $1=\frac{10S}{9}-\frac{S}{9}=\frac{9S}{9}$.
Whence $S=1=$ *ſum required.*

6. Required the ſum (S) of the ſeries $a^2+\overline{a+d}|^2+\overline{a+2d}|^2+\overline{a+3d}|^2+\overline{a+4d}|^2$, &c. continued to n terms.

Firſt, $a^2 = a^2$

$\overline{a+d}|^2 = a^2 + 2 \times 1ad + 1d^2$

$\overline{a+2d}|^2 = a^2 + 2 \times 2ad + 4d^2$

$\overline{a+3d}|^2 = a^2 + 2 \times 3ad + 9d^2$

$\overline{a+4d}|^2 = a^2 + 2 \times 4ad + 16d^2$

&c. &c.

Therefore, $S = \begin{cases} \text{Sum of } n \text{ terms of } \overline{1+1+1+1, \&c.} \\ \times a^2 \\ + \ldots \text{ ditto of } \overline{0+1+2+3, \&c.} \\ \times 2ad \\ + \ldots \text{ ditto of } \overline{0+1+4+9, \&c.} \\ \times d^2 \end{cases}$

But n terms of $1+1+1+1$, &c. $= n$

Ditto of $0+1+2+3$, &c. $= \frac{n \times \overline{n-1}}{1 \times 2}$

And ditto of $0+1+4+9$, &c. $= \frac{n \times \overline{n-1} \times \overline{2n-1}}{1 \times 2 \times 3}$

Whence $S = n \times a^2 + n \times \overline{n-1} \times ad + \frac{n \times \overline{n-1} \times \overline{2n-1}}{1 \times 2 \times 3} \times d^2 = na^2 + nad \times \overline{n-1} + \frac{nd^2 \times \overline{n-1} \times \overline{2n-1}}{1 \times 2 \times 3} =$ *ſum required.*

7. Required the ſum (S) of the ſeries $a^3 + \overline{a+d}|^3 + \overline{a+2d}|^3 + \overline{a+3d}|^3 + \overline{a+4d}|^3$, &c. continued to n terms.

Firſt, $a^3 = a^3$

$\overline{a+d}|^3 = a^3 + 3 \times 1a^2d + 3 \times 1ad^2 + 1d^3$

$\overline{a+2d}|^3 = a^3 + 3 \times 2a^2d + 3 \times 4ad^2 + 8d^3$

$\overline{a+3d}|^3 = a^3 + 3 \times 3a^2d + 3 \times 9ad^2 + 27d^3$

$\overline{a+4d}|^3 = a^3 + 3 \times 4a^2d + 3 \times 16ad^2 + 64d^3$

Therefore, $$S=\left\{\begin{array}{l}\textit{Sum of } n \textit{ terms of } \overline{1+1+1+1,\ \&c.}\\ \quad\times a^3\\ +\ \ldots\ \textit{ditto of } \overline{0+1+2+3,\ \&c.}\\ \quad\times 3a^2d\\ +\ \ldots\ \textit{ditto of } \overline{0+1+4+9,\ \&c.}\\ \quad\times 3ad^2\\ +\ \ldots\ \textit{ditto of } \overline{0+1+8+27,\ \&c.}\\ \quad\times d^3\end{array}\right.$$

But n terms of $1+1+1+1$, *&c.* $=n$

Ditto . . . of $0+1+2+3$, *&c.* $=\dfrac{n\times\overline{n-1}}{1\times2}$

Ditto . . . of $0+1+4+9$, *&c.* $=\dfrac{n\times\overline{n-1}\times\overline{2n-1}}{1\times2\times3}$

Ditto . . . of $0+1+8+27$, *&c.* $=\dfrac{n^4-2n^3+n^2}{2\times2}$

Whence $S=na^3+\dfrac{n\times\overline{n-1}\times3a^2d}{1\times2}+\dfrac{n\times\overline{n-1}\times\overline{2n-1}\times3ad^2}{1\times2\times3}+\dfrac{\overline{n^4-2n^3+n^2}\times d^3}{2\times2}=$ *ſum required.*

8. Required the ſum (S) of n terms of the ſeries $1+3+7+15+31$, &c.

The terms of this ſeries are evidently equal to 1, $\overline{1+2}$, $\overline{1+2+4}$, $\overline{1+2+4+8}$, *&c. or the ſucceſſive ſums of the geometrical progreſſion* 1, 2, 4, 8, 16, *&c.*

Let, therefore, $a=1$, *and* $r=2$, *and we ſhall have* $a+ar+ar^2+ar^3+ar^4$, *&c.* $=1+2+4+8+16$, *&c.*

But the ſums of 1, 2, 3, 4, *&c. terms of this ſeries, are*

$$\left|\begin{array}{l|l}
1 & \frac{ar-a}{r-1} = \overline{r-1} \times \frac{a}{r-1} \\
2 & \frac{ar^2-a}{r-1} = \overline{r^2-1} \times \frac{a}{r-1} \\
3 & \frac{ar^3-a}{r-1} = \overline{r^3-1} \times \frac{a}{r-1} \\
4 & \frac{ar^4-a}{r-1} = \overline{r^4-1} \times \frac{a}{r-1} \\
 & \text{\&c.} \qquad\qquad \text{\&c.}
\end{array}\right.$$

Therefore, $S = \frac{a}{r-1} \times \left\{\begin{array}{l} n \text{ terms of } r + r^2 + r^3 + r^4, \text{ \&c.} \\ - n \text{ terms of } 1 + 1 + 1 + 1, \text{ \&c.} \end{array}\right.$

But $1 + 1 + 1 + 1$, *&c.* $= n$

And $r + r^2 + r^3 + r^4$, *&c.* $= \overline{r^n - 1} \times \frac{r}{r-1}$.

Whence $S = \overline{\overline{r^n - 1} \times \frac{r}{r-1} - n} \times \frac{a}{r-1} =$ *ſum required.*

9. Required the ſum (S) of n terms of the ſeries $\frac{1}{1} + \frac{3}{2} + \frac{7}{4} + \frac{15}{8} + \frac{31}{16}$, &c. the terms of which are the ſucceſſive ſums of the geometrical progreſſion $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$, &c.

Let $a = 1$ *and* $r = 2$, *then will* $a + \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3} + \frac{a}{r^4}$, *&c.* $= \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$, *&c.*

But the ſums of 1, 2, 3, 4, *&c. terms of this ſeries are*

$$\begin{array}{l|l}
1 & \dfrac{\overline{r-1}\times a}{\overline{r-1}\times 1}=\overline{r-1}\times\dfrac{a}{r-1} \\
2 & \dfrac{\overline{r^2-1}\times a}{\overline{r-1}\times r}=\overline{r-\dfrac{1}{r}}\times\dfrac{a}{r-1} \\
3 & \dfrac{\overline{r^3-1}\times a}{\overline{r-1}\times r^2}=\overline{r-\dfrac{1}{r^2}}\times\dfrac{a}{r-1} \\
4 & \dfrac{\overline{r^4-1}\times a}{\overline{r-1}\times r^3}=\overline{r-\dfrac{1}{r^3}}\times\dfrac{a}{r-1} \\
 & \text{\&c.} \qquad\qquad \text{\&c.}
\end{array}$$

Therefore $S=\dfrac{a}{r-1}\times\left\{\begin{array}{l} n \text{ terms of } r+r+r+r, \text{ \&c.} \\ -n \text{ terms of } \dfrac{1}{1}+\dfrac{1}{r}+\dfrac{1}{r^2}+\dfrac{1}{r^3}, \text{ \&c.}\end{array}\right.$

But $r+r+r+r$, *&c.* $= nr$,

And $\dfrac{1}{1}+\dfrac{1}{r}+\dfrac{1}{r^2}+\dfrac{1}{r^3}$ *&c.* $=\dfrac{r^n-1}{\overline{r-1}\times r^{n-1}}$,

Whence $S=\dfrac{a}{r-1}\times\overline{nr-\dfrac{r^n-1}{\overline{r-1}\times r^{n-1}}}=$ *ſum required.*

10. To find the ſum (S) of the infinite ſeries of the reciprocals of the triangular numbers, $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$, &c.

Let $\dfrac{1}{1}+\dfrac{1}{3}+\dfrac{1}{6}+\dfrac{1}{10}$, *&c. ad infinitum* $= S$.

Or, $\dfrac{1}{1.1}+\dfrac{1}{1.3}+\dfrac{1}{2.3}+\dfrac{1}{2.5}$, *&c.* . . . . $= S$.

Then $\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+\dfrac{1}{4.5}$ *&c.* . . . . $=\dfrac{S}{2}$.

That is, $\dfrac{1}{1}-\dfrac{1}{2}+\dfrac{1}{2}-\dfrac{1}{3}+\dfrac{1}{3}-\dfrac{1}{4}+\dfrac{1}{4}-\dfrac{1}{5}$, *&c.* $=\dfrac{S}{2}$.

$$\text{Or,}\ \left\{\begin{array}{l}\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\ \text{\&c.}\\ \quad-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}\ \text{\&c.}\end{array}\right\}=\frac{S}{2}.$$

$$\text{Whence, }\frac{S}{2}=\frac{1}{1};\ \text{or } S=2=\text{ſum required.}$$

* 11. To find the ſum of n terms of the ſeries $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c.

$$\text{Let } z=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5},\ \text{\&c. to } \frac{1}{n}.$$

$$\text{Then } z-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\ \text{\&c. to } \frac{1}{n}.$$

$$\text{And } z-\frac{1}{1}+\frac{1}{n+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\ \text{\&c. to } \frac{1}{n+1}$$

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* Let Σ = ſum of an infinite number of terms, and S = ſum of n terms.

Then the *formulæ* for the ſums of the reciprocals of figurate numbers may be exhibited as follows:

$$\begin{array}{c|lcl}
 & & \Sigma & S \\
1 & \frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1},\ \text{\&c.}= & \infty & =n \\
2 & \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4},\ \text{\&c.}= & \infty & = \\
3 & \frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10},\ \text{\&c.}= & \frac{2}{1} & =\frac{2}{1}-\frac{2}{1}\times\frac{1}{n+1} \\
4 & \frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20},\ \text{\&c.}= & \frac{3}{2} & =\frac{3}{2}-\frac{3}{2}\times\frac{1.2}{n-1.n-2} \\
5 & \frac{1}{1}+\frac{1}{5}+\frac{1}{15}+\frac{1}{35},\ \text{\&c.}= & \frac{4}{3} & =\frac{4}{3}-\frac{4}{3}\times\frac{1.2.3}{n+1.n+2.n+3} \\
 & \quad\text{\&c.} & \text{\&c.} & \quad\text{\&c.}
\end{array}$$

Therefore $\frac{1}{1}-\frac{1}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$ *&c. to* $\frac{1}{n}-\frac{1}{n+1}$

Or, $\frac{n}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$ *&c. to* $\frac{1}{n.\overline{n+1}}$.

Whence, $\frac{2n}{n+1}=\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$ *&c. to* $\frac{2}{n.\overline{n+1}}$.

Or, $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$ *&c. to* $\frac{2}{n.\overline{n+1}}$ (*or n terms*) $=\frac{2n}{n+1}=$ *ſum required.*

12. Required the ſum of the infinite ſeries $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, &c.

Let $z=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, *&c. ad infinitum.*

Then $z-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, *&c. by tranſpoſition.*

And $1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}$, *&c. by ſubtraction.*

Or $1-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}$, *&c. by tranſpoſition.*

And $\frac{1}{2}=\frac{4}{1.4.3}+\frac{6}{2.9.4}+\frac{8}{3.16.5}$, *&c. by ſubtraction.*

Or $\frac{1}{2}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+\frac{2}{4.5.6}$, *&c.*

Whence $\frac{1}{2}\div 2=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, *&c.*

Or, $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, *&c.* $=\frac{1}{2}\div 2=\frac{1}{4}=$ *ſum required.*

13. To find the ſum of *n* terms of the ſeries $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6}$, &c.

Let $z = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} +$, *&c. to* $\frac{1}{n.\overline{n+1}}$.

Then $z - \frac{1}{2} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}$, *&c. to* $\frac{1}{n.\overline{n+1}}$.

And $z - \frac{1}{2} + \frac{1}{\overline{n+1}.\overline{n+2}} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}$, *&c. to* $\frac{1}{\overline{n+1}.\overline{n+2}}$.

Therefore $\frac{1}{2} - \frac{1}{\overline{n+1}.\overline{n+2}} = \frac{2}{1.2.3} + \frac{2}{2.3.4} + \frac{2}{3.4.5}$, *&c.* (*continued to n terms*) *by ſubtraction.*

Whence $\frac{1}{4} - \frac{1}{2.\overline{n+1}.\overline{n+2}} = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}$, *&c.* (*continued to n terms*) *by diviſion.*

And conſequently, $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}$, *&c. to n terms* $= \frac{1}{4} - \frac{1}{2.\overline{n+1}.\overline{n+2}} =$ *ſum required.*

14. Required the ſum (S) of the ſeries $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16}$, &c. continued *ad infinitum.*

Let $x = \frac{1}{2}$ *and* $S = \frac{x}{1+x}$.

Then $\frac{x}{1+x} = x - x^2 + x^3 - x^4 + x^5$, *&c.*

And $x = \overline{1+x} \times \overline{x - x^2 + x^3 - x^4, \&c.}$

$$=\left\{\begin{array}{l} x-x^2+x^3-x^4+x^5,\ \text{\&c.} \\ 1+x \end{array}\right.$$

$$\begin{array}{r} x-x^2+x^3-x^4+x^5,\ \text{\&c.} \\ +x^2-x^3+x^4-x^5,\ \text{\&c.} \\ \hline \end{array}$$

$$=x+0+0+0+0,\ \text{\&c.}$$

Therefore $z=x$.

And $x-x^2+x^3-x^4+x^5$, &c. $=\frac{x}{1+x}$.

Or $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}$, &c. $=\frac{\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{3}=$ *ſum required.*

15. Required the ſum of the infinite ſeries $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}$, &c.

Let $x=\frac{1}{2}$, *and* $S=\frac{z}{\overline{1-x}|^2}$.

Then $\frac{z}{\overline{1-x}|^2}=x+2x^2+3x^3+4x^4+5x^5$, &c.

And $z=\overline{1-x}|^2\times\overline{x+2x^2+3x^3+4x^4,\ \text{\&c.}}$

$$=\left\{\begin{array}{l} x+2x^2+3x^3+4x^4,\ \text{\&c.} \\ 1-2x+x^2 \end{array}\right.$$

$$\begin{array}{r} x+2x^2+3x^3+4x^4,\ \text{\&c.} \\ -2x^2-4x^3-6x^3,\ \text{\&c.} \\ +x^3+2x^4,\ \text{\&c.} \\ \hline x+0+0+0,\ \text{\&c.} \end{array}$$

Therefore $x=z$.

And $x+2x^2+3x^3+4x^4+5x^5$, &c. $=\frac{x}{\overline{1-x}|^2}$.

Or $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}$, &c. $=\frac{\frac{1}{2}}{\overline{1-\frac{1}{2}}|^{2}}=2=$ *ſum required.*

16. Required the ſum (S) of the infinite ſeries $\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}$, &c.

Let $x=\frac{1}{3}$, *and* $\frac{z}{\overline{1-x}|^{3}}=S$.

Then $\frac{z}{\overline{1-x}|^{3}}=x+4x^{2}+9x^{3}+16x^{4}+25x^{5}$, *&c.*

And $z=\overline{1-x}|^{3}\times\overline{x+4x^{2}+9x^{3}+16x^{4},\ \&c.}$
$=x+x^{2}$, *as will be found by actual multiplication.*
Therefore $x+x^{2}=z$.

And conſequently $x+4x^{2}+9x^{3}+16x^{4}$, *&c.* $=\frac{x\times\overline{1+x}}{\overline{1-x}|^{3}}$

Or $\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}$, *&c.* $=\frac{\frac{1}{3}\times\overline{1+\frac{1}{3}}}{\overline{1-\frac{1}{3}}|^{3}}=\frac{3}{2}=$ *ſum required.*

17. Required the ſum (S) of the infinite ſeries $\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^{2}}+\frac{a+3d}{mr^{3}}$, &c.

Let $x=\frac{1}{r}$, *and* $S=\frac{z}{m.\overline{1-x}|^{2}}$.

Then $\frac{z}{mr\,\overline{1-x}|^{2}}=\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^{2}}+\frac{a+3d}{mr^{3}}$, *&c.*

Or $\frac{z}{\overline{1-x}|^{2}}=a+\frac{a+d}{r}+\frac{a+2d}{r^{2}}+\frac{a+3d}{r^{3}}$, *&c.*

That is $\frac{z}{\overline{1-x}|^{2}}=a+\overline{a+d}\times x+\overline{a+2d}\times x^{2}+\overline{a+3d}\times x^{3}$, *&c.*

And $z = \overline{1-x}|^2 \times \overline{a + \overline{a+d}x + \overline{a+2d}x^2 + \overline{a+3d}x^3, \&c.}$ $= \overline{1-x} \times a + dx$, *as will appear by actual multiplication.*

Therefore $z = \overline{1-x} \times a + dx$.

And conſequently $\frac{a}{m} + \frac{a+d}{mr} + \frac{a+2d}{mr^2}$, *&c.* $=$ $\frac{\overline{1-x} \times a + dx}{m . \overline{1-x}|^2}$ $=$ *ſum required.*

EXAMPLES FOR PRACTICE.

1. To find the ſum of n terms of the ſeries $a + \overline{a-d} + \overline{a-2d} + \overline{a-3d} + \overline{a-4d}$, &c.

Anſ. $\frac{n}{2} \times \overline{2a - \overline{n-1} \times d}$.

2. Required the ſum of the infinite ſeries $a + da + d^2a + d^3a + d^4a$, &c. where d is a proper fraction.

Anſ. $\frac{a}{1-d}$.

3. To find the ſum of the infinite ſeries $1 + 2^4x + 3^4x^2 + 4^4x^3 + 5^4x^4$, &c. *Anſ.* $\frac{1 + 11x + 11x^2 + x^3.}{\overline{1-x}|^6}$

4. Required the ſum of the infinite ſeries $1 + 3x + 6x^2 + 10x^3 + 15x^4$, &c. *Anſ.* $\frac{1}{\overline{1-x}|^2}$.

5. Required the ſum of the infinite ſeries $1 + 4x + 10x^2 + 20x^3 + 35x^4$, &c. *Anſ.* $\frac{1}{\overline{1-x}|^3}$.

6. Required the ſum of the infinite ſeries $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9}$, &c. *Anſ.* $\frac{1}{2}$.

7. Required the ſum of 40 terms of the ſeries $\overline{1\times2}+\overline{3\times4}+\overline{5\times6}+\overline{6\times7}$, &c. Anſ. 22960.

8. Required the ſum of the infinite ſeries $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}$, &c. Anſ. $\frac{1}{18}$.

9. Required the ſum of n terms of the ſeries $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}$, &c.

Anſ. $\frac{1}{18}-\frac{1}{3.\overline{n+1}.\overline{n+2}.\overline{n+3}}$.

10. Required the ſum of n terms of the ſeries $\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}$, &c.

Anſ. $\frac{\overline{1-p}\times r}{\overline{r-1}|^2}-\frac{np}{r-1}$, *where* $p=\frac{1}{r}$.

11. Required the ſum of the ſeries $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}$, &c. $\frac{1}{n\times\overline{3+n}}$.

Anſ. $\Sigma=\frac{11}{18}$, $S=\frac{n}{3+3n}+\frac{n}{12+6n}-\frac{n}{27+9n}$.

12. Required the ſum of the ſeries $\frac{1}{2.6}+\frac{1}{4.8}+\frac{1}{6.10}$, &c. $\frac{1}{2n\,\overline{4+2n}}$.

Anſ. $\Sigma=\frac{3}{16}$, $S=\frac{5n+3n^2}{32+48n+16n^2}$.

13. Required the ſum of the ſeries $\frac{1}{4.8} - \frac{1}{6.10} + \frac{1}{8.12} -$, &c. $\frac{1}{\overline{2+2n}.\overline{6+2n}}$.

$$Anſ.\ \Sigma = \frac{1}{48},\ S = \frac{n}{16+16n} - \frac{n}{36+24n}.$$

14. Required the ſum of the ſeries $\frac{1}{3.8} + \frac{1}{6.12} + \frac{1}{9.16} + \frac{1}{12.20}$, &c. $\frac{1}{3n.\overline{4+4n}}$.

$$Anſ.\ \Sigma = \frac{1}{12},\ S = \frac{n}{12+12n}.$$

* 15. Required the ſum of the ſeries $\frac{1}{2.6.4.5} + \frac{1}{4.8.5.6} + \frac{1}{6.10.6.7}$, &c. $\frac{1}{2n.\overline{4+2n}.\overline{3+n}.\overline{4+n}}$.

$$Anſ.\ \Sigma = \frac{7}{1152},\ S = \frac{n}{24} \times : \frac{1}{4.\overline{1+n}} + \frac{1}{8.\overline{2+n}} - \frac{5}{12.\overline{3+n}} + \frac{3}{16.\overline{4+n}}.$$

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* A great variety of ſeries, of different forms, may be found in other authors; but thoſe which are here given will be ſufficient for the learner's practice.

The names of the principal authors, who have written upon this ſubject, are as follow:

Archimedes; Arabes; D' Alembert; Barrow; Briggs; Nicholas, Daniel, John and James Bernoulli; Fermat; De Cartes; Clairaut; Condercat; Cotes; Dodſon; Euler; Emerſon; Fagnanus; Le Grange; Goldbach; Gregory; Halley; Harriot; Huddens; Huygens; Hutton; Kepler; Keil; Landen; Mac Laurin; De Lagney; Leibnitz; Lorgna; Lucas de Burgo; Manfredi; Monmort; De Moivre; Montono; Nichole; Newton; Oughtred; Riccati; Regnald; Saunderſon; Sterling; Sluſius; Simpſon; Brook Taylor; Varignon; Vieta; Wallis; Waring; &c.

OF LOGARITHMS*.

Logarithms are numbers ſo contrived and adapted to other numbers, that the ſums and differences of the former ſhall correſpond to, and ſhew, the products and quotients of the latter.

Or, more generally, logarithms are the numerical exponents of ratios; or a ſeries of numbers in arithmetical progreſſion, anſwering to another ſeries of numbers in geometrical progreſſion.

Thus { 0. 1. 2. 3. 4. 5. *Indices, or logarithms.*
{ 1. 2. 4. 8. 16. 32. *Geometric progreſſion.*
Or { 0. 1. 2. 3. 4. 5. *Indices, or logarithms.*
{ 1. 3. 9. 27. 81. 243. *Geometric progreſſion.*
Or { 0. 1. 2. 3. 4. 5. *Ind. or log.*
{ 1. 10. 100. 1000. 10000. 100000. *Geo. prog.*
&c.

And, from hence, it is evident, that there may be as many kinds of indices, or logarithms, as there can be taken different kinds of geometric ſeries; and that in any ſyſtem, or table, of logarithms whatever, the logarithm of unity, or 1, will be nothing.

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* The invention of logarithms is the undoubted right of Lord *Neper*, Baron of *Merchiſton*, in *Scotland*, and is properly conſidered as one of the moſt uſeful and excellent diſcoveries of modern times. A table of theſe numbers was firſt publiſhed by him at Edinburgh, anno 1614, in a treatiſe entitled *Canon Mirificum Logarithmorum*; and as their great utility and extenſive application, were ſufficiently apparent, they were immediately received by all the learned throughout Europe. Mr. *Henry Briggs*, *Savilian* Profeſſor of Geometry at *Oxford*,

It is, alſo, apparent, from the nature of theſe ſeries, that, if any two indices be added together, their ſum will be the index of that number which is equal to the product of the two terms, in the geometic progreſſion, to which thoſe indices belong.

Thus, the indices 2 and 3, being added together, are =5; and the numbers 4 and 8, or the terms correſponding with thoſe indices, being multiplied together, are =32, which is the number anſwering to the index 5.

And, in like manner, if any one index be ſubtracted from another, the difference will be the index of that number which is equal to the quotient of the two terms to which thoſe indices belong.

Thus the index 6, minus the index 4, is =2; and the terms correſponding to thoſe indices are 64 and 16, whoſe quotient is =4; which is the number anſwering to the index 2.

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upon hearing of the diſcovery, ſet out on a viſit to the noble inventor, and ſoon afterwards they jointly undertook the arduous taſk of computing new tables upon this ſubject, and reducing them to a more convenient form than that which was at firſt thought of. But Lord *Neper* dying before they were finiſhed, the whole burden was laid upon Mr. *Briggs*, who with prodigious labour, and great ſkill, made an entire *Canon*, according to the new form, for all numbers from 1 to 20000, and from 90000 to 101000, to 14 places of figures, and publiſhed it at *London* in the year 1624, in a treatiſe entitled *Arithmetica Logarithmica*, with directions for ſupplying the intermediate *chiliads*.

This *Canon* was again publiſhed in *Holland*, by *Adrian Vlacq*, *anno* 1628, together with the logarithms of all the numbers which Mr. *Briggs* had omitted; but he continued them only to 10 places of decimals. Mr. *Briggs* alſo computed the logarithms of the ſigns, tangents and ſecants, to every degree, and $\frac{1}{100}$ part of a degree of the whole quadrant; and ſubjoined

For the ſame reaſon, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power.

Thus, the index, or logarithm of 4, *in the above ſeries, is* 2; *and if this number be multiplied by* 3, *the product will be* $=6$; *which is the logarithm of* 64, *or the third power of* 4.

And, if the logarithm of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root.

Thus the index, or logarithm of 64 *is* 6; *and if this number be divided by* 2, *the quotient will be* $=3$; *which is the logarithm of* 8, *or the ſquare root of* 64.

The logarithms moſt convenient for practice are ſuch as are adopted to a geometric ſeries increaſing

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them to the natural ſigns, tangents and ſecants, which he had before computed to 15 places of figures. And theſe tables, together with their conſtruction and uſe, were firſt publiſhed in the year 1633, after Mr. *Briggs*'s death, by Mr. *Henry Gellibrand*, under the title of *Trigonometria Britannica.*

Benjamin Urſinus has alſo given us a table of logarithms to every 10 ſeconds. And Mr. *Wolf*, in his *Mathematical Lexicon*, ſays, that one *Van Loſer* had computed them to every ſingle ſecond; but his untimely death prevented their publication.

A great number of other authors have treated of this ſubject, but as their numbers are frequently inaccurate, and incommodiouſly diſpoſed, they are now generally neglected. The tables in moſt repute at preſent, are thoſe of *Gardiner* in 4to, firſt printed in the year 1742, and *Sherwin* in 8vo, firſt printed in the year 1705, where the logarithms of all numbers may be eaſily found from 1 to 10000000; and thoſe of the ſigns, tangents, and ſecants, to any degree of accuracy required.

Dodſon's *Antilogarithmic Canon* is likewiſe a very ingenious work, and is of great uſe for finding the numbers anſwering to any given logarithms.

in a tenfold proportion, as in the laſt of the above examples; and are thoſe which are to be found, at preſent, in moſt of the common tables upon this ſubject.

And the diſtinguiſhing mark of this ſyſtem of logarithms is, that the index, or logarithm, of 1 is 0; that of 10, 1; that of 100, 2; that of 1000, 3, &c. And, from hence it follows, that the logarithm of any number between 1 and 10 muſt be 0 and ſome fractional parts; and that of a number between 10 and 100 will be 1 and ſome fractional parts; and ſo on for any other number whatever.

And ſince the integral part of a logarithm is always thus readily found, it is uſually called the *index*, or *characteriſtic*; and is commonly omitted in the tables; being left to be ſupplied by the operator himſelf, as occaſion requires.

OF THE MAKING OF LOGARITHMS.

Whatever arithmetical progreſſion we apply to a geometrical one, the terms of it are logarithms only to that ſeries to which we apply them, and anſwer the end propoſed only for thoſe particular numbers; ſo that if we had logarithms adapted only to particular geometrical ſeries, they would be but of little uſe. The great end and deſign of logarithms is the eaſe and expedition they afford in long calculations, by ſaving the laborious work of *multiplication*, *diviſion*, and the *extraction of roots*; but this end would never be completely anſwered, unleſs logarithms could be adapted to the whole ſyſtem of numbers, 1, 2, 3, 4, &c. And as here lay the excellence and merit of the contrivance, ſo alſo the difficulty. For the natural ſyſtem of numbers, 1, 2, 3, 4, &c. being an arithmetical, and not a geometrical ſeries, ſeems rather fit to be made logarithms of, than to have logarithms applied to it. Yet this difficulty may be eaſily removed, by conſidering,

That though the whole ſyſtem of natural numbers, 1, 2, 3, 4, &c. makes not one geometrical ſeries, and cannot, by any means, be brought within ſuch a ſeries of determinate numbers, yet they may be brought ſo near to it, as to be within any aſſignable degree of approximation; which may be conceived, in general, thus: Suppoſe a fraction indefinitely ſmall to be repreſented by x, and a geometrical ſeries ariſing from 1, in the ratio of 1 to $1+x$, to be 1, $\overline{1+x}|^{1}$, $\overline{1+x}|^{2}$, $\overline{1+x}|^{3}$, $\overline{1+x}|^{4}$, &c. Then muſt ſome of theſe terms come indefinitely near to coincide with all the natural numbers, 1, 2, 3, 4, &c.; becauſe amongſt numbers that ariſe by indefinitely ſmall increments, ſome of them muſt exceed, or fall ſhort, of any determinate number, by an indefinitely little exceſs or defect.

If, therefore, in the places of the terms of this ſeries, that do approach indefinitely near to any of the natural numbers, we ſuppoſe theſe natural numbers themſelves to be ſubſtituted, then will the ſeries be a geometrical progreſſion, to an exactneſs that may be called *indefinite*; becauſe the approximation of its terms to the natural numbers, can never end, but goes on *in infinitum*.

And ſince this imagined geometric ſeries comprehends, indefinitely near, the whole ſyſtem of natural numbers, 1, 2, 3, 4, &c. ſo the indices of its terms comprehend a whole ſyſtem of logarithms, which are adapted to this ſyſtem of numbers, and may be extended to any length we pleaſe. For though the natural ſyſtem of numbers make not, by themſelves, a complete geometrical ſeries, yet they are conceived as a part of ſuch a ſeries, and their logarithms are the indices of their diſtances from unity in that ſeries; or, more generally, they are the correſponding terms of an arithmetical ſeries applied to that geometrical one.

But, again, it muſt be obſerved, that an indefinitely ſmall fraction cannot be aſſigned; and, therefore, in the actual conſtruction of logarithms, we muſt be contented with a determinate degree of approximation. Whence, accordingly as we take x, ſo in the ſeries 1, $\overline{1+x}|^{1}$, $\overline{1+x}|^{2}$, $\overline{1+x}|^{3}$, $\overline{1+x}|^{4}$, &c. the approximation of its terms to the natural numbers will be in different degrees of exactneſs: for the leſs x is, the nearer will be the approximation; but then the more are the number of involutions of $1+x$, neceſſary to come within any determinate degree of nearneſs to the natural number aſſigned.

Thus then we may conceive the poſſibility of making logarithms to all the natural numbers, 1, 2, 3, 4, &c. to any determinate degree of exactneſs; viz. by aſſigning a very ſmall fraction for x, and actually raiſing a ſeries, in the ratio of 1 to $1+x$, and taking for the natural numbers ſuch terms of that ſeries as are the neareſt to them, and their indices for the logarithms. But then, to conſtruct logarithms in this manner, to ſuch an extent of numbers, and degree of exactneſs, as would be neceſſary to make them of any conſiderable uſe, is next to impoſſible, becauſe of the almoſt infinite labour and time it would require. This, however, is an introduction for underſtanding the method of the *noble inventor*, who undoubtedly firſt took the hint of making logarithms from the conſideration of the indices of a geometrical ſeries; and by means of the principles and known properties of theſe progreſſions, he firſt formed his tables, and adapted them to the practical purpoſes intended.

PROBLEM I.

To find the logarithm of any of the natural numbers, 1, 2, 3, 4, *&c. according to the method of* NEPER.

RULE.

1. Take the geometrical ſeries, 1, 10, 100, 1000, 10000, &c. and apply to it the arithmetical ſeries 1, 2, 3, 4, 5, &c, as logarithms.

2. Find a geometric mean between 1 and 10, 10 and 100, or any other two adjacent terms of the ſeries betwixt which the number propoſed lies.

3. Between the mean, thus found, and the neareſt extreme, find another geometrical mean, in the ſame manner; and ſo on, till you are arrived within the propoſed limit of the number whoſe logarithm is ſought.

4. Find as many arithmetical means, in the ſame order as you found the geometrical ones, and the laſt of theſe will be the logarithm anſwering to the number required.

EXAMPLE:

Let it be required to find the logarithm of 9.

Here the numbers between which 9 *lies are* 1 *and* 10.

Firſt, then, the log. of 10 *is* 1, *and the log. of* 1 *is* 0; *therefore* $\frac{1+0}{2} = .5$ *is the arithmetical mean. And* $\sqrt{1 \times 10} = \sqrt{10} = 3.1622777$ = *geometric mean*; *whence the logarithm of* 3.1622777 *is* .5.

Secondly, the log. of 10 *is* 1, *and the log. of* 3.1622777 *is* .5; *therefore* $\frac{1+.5}{2} = .75 =$ *arithmetical mean. And* $\sqrt{10 \times 3.1622777} = 5.6234132 =$ *geometric mean; whence the log. of* 5.6234132 *is* .75.

Thirdly, the log. of 10 *is* 1, *and the log. of* 5.6234132 *is* .75; *therefore* $\frac{1+.75}{2} = .875 =$ *arithmetical mean. And* $\sqrt{10. \times 5.6234132} = 7.4989421$ $=$ *geometric mean; whence the log. of* 7.4989421 *is* .875.

Fourthly, the log. of 10 *is* 1, *and the log. of* 7.4989421 *is* .875; *therefore* $\frac{1+.875}{2} = .9375 =$ *arithmeticalmean. And* $\sqrt{10 \times 7.4989421} = 8.6596431$ $=$ *geometric mean; whence the log. of* 8.6596431 *is* .9375.

Fifthly, the log. of 10 *is* 1, *and the log. of* 8.6596431 *is* .9375; *therefore* $\frac{1+.9375}{2} = .96875$ $=$ *arithmetical mean. And* $\sqrt{10 \times 8.6596431} =$ 9.3057204 $=$ *geometric mean; whence the log. of* 9.3057204 *is* .96875.

Sixthly, the log. of 8.6596431 *is* .9375, *and the log. of* 9.3057204 *is* .96875; *therefore* $\frac{.9375+.96875}{2} =$ .953125 $=$ *arithmetical mean. And* $\sqrt{8.6596431 \times 9.3057204} = 8.9768713 =$ *geometric mean; whence the log. of* 8.9768713 *is* .953125.

And, proceeding in this manner, after 25 *extractions, the logarithm of* 8.9999998 *will be found to be* .9542425; *which may be taken for the logarithm of* 9, *becauſe it differs from it only by* $\frac{1}{5000000}$, *and is therefore ſufficiently exact for all practical purpoſes.*

And in the ſame manner the logarithms of almoſt all the prime numbers were found; a work ſo incredibly laborious, that the unremitted induſtry of ſeveral years was ſcarcely ſufficient for its accompliſhment.

PROBLEM II.

To determine the hyperbolic logarithm (L) of any given number (N).

The hyperbolic logarithm of any number, is the index of that term of the logarithmic progreſſion, agreeing with the propoſed number, multiplied by the exceſs of the common ratio above unity.

Let, therefore, $\overline{1+x}|^{n}$ be that term of the logarithmic progreſſion, 1, $\overline{1+x}^{1}$, $\overline{1+x}|^{2}$, $\overline{1+x}|^{3}$, $\overline{1+x}|^{4}$, &c. which is equal to the required number (N).

Then will $\overline{1+x}|^{n}=N$, and $1+x=N^{\frac{1}{n}}$; and if $1+y$ be put $=N$, and $m=\frac{1}{n}$, we ſhall have $1+x=N^{\frac{1}{n}}=\overline{1+y}|^{n}=1+my+m\times\frac{m-1}{2}y^{2}+m\times\frac{m-1}{2}\times\frac{m-2}{3}y^{3}$, &c.

And, conſequently, $x=my+m\times\frac{m-1}{2}y^{2}+m\times\frac{m-1}{2}\times\frac{m-2}{3}y^{3}$, &c. where m being rejected in the factors $m-1$, $m-2$, $m-3$, &c. as being indefinitely ſmall in compariſon of 1, 2, 3, &c. the equation will become $x=my-\frac{my^{2}}{2}+\frac{my^{3}}{3}-\frac{my^{4}}{4}$, &c.

Hence $\frac{x}{m}$ $(nx=L)=x-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}$, &c. = hyperbolic logarithm of N, as was required.

PROBLEM III.

The hyperbolic logarithm (L) *of a number being given, to find the number* (N) *itſelf, anſwering thereto.*

Let $\overline{1+x}|^{n}$ be that term of the logarithmic progreſſion, 1, $\overline{1+x}|^{1}$, $\overline{1+x}|^{2}$, $\overline{1+x}|^{3}$, $\overline{1+x}|^{4}$, &c. which is equal to the required number N.

Then, becauſe $\overline{1+x}|^{n}$ is univerſally $=1+nx+n\times\frac{n-1}{2}x^2+n\times\frac{n-1}{2}\times\frac{n-2}{3}x^3$, &c. we ſhall alſo have

$$1+nx+n\times\frac{n-1}{2}x^2+n\times\frac{n-1}{2}\times\frac{n-2}{3}x^3, \text{ \&c.} = N.$$

But becauſe n, from the nature of logarithms, is here ſuppoſed indefinitely great, it is evident that the numbers connected to it by the ſign —, may all be rejected, as far as any aſſigned number of terms, being indefinitely ſmall in compariſon of n.

Therefore, by throwing out 1, 2, 3, &c. from the factors $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, &c. we ſhall have $1+nx+\frac{n^2x^2}{2}+\frac{n^3x^3}{2.3}+\frac{n^4x^4}{2.3.4}$, &c. $=N$.

But nx $(=L)$ is the hyperbolic logarithm of $\overline{1+x}|^{n}$, or N, by what has been before ſpecified; and therefore $1+L+\frac{L^2}{2}+\frac{L^3}{2.3}+\frac{L^4}{2.3.4}$, &c. $=N=$ number required.

PROBLEM IV.

To determine the hyperbolic logarithm (L) *of any given number* (N), *by an univerſally converging ſeries.*

The ſeries $y - \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4}$, &c. is the moſt eaſy and natural that can be obtained; but, in determining the logarithms of large numbers, it is but of little uſe, ſince, in all ſuch caſes, it diverges inſtead of converging.

Let, therefore, the number whoſe logarithm you would find, be denoted by $\frac{1}{1-y}$, and alſo let $\overline{1+x}|^n$ be the term of the logarithmic progreſſion agreeing with the propoſed number.

Then $\overline{1+x}|^n = \frac{1}{1-y}$; or $1+x = \frac{1}{\overline{1-y}|^{\frac{1}{n}}} = \overline{1-y}|^{-\frac{1}{n}} = \overline{1-y}|^m$ (by putting $m = -\frac{1}{n}$) $= 1 - my + m \times \frac{m-1}{2} y^2 - m \times \frac{m-1}{2} \times \frac{m-2}{3} y^3$, &c.

And, if m be rejected in the factors $m-1$, $m-2$, $m-3$, as before, our equation will become $1+x = 1 - my - \frac{my^2}{2} - \frac{my^3}{3} - \frac{my^4}{4}$ &c.

Whence $y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4}$ &c. $= -\frac{x}{m} = nx =$ hyperbolic logarithm of $\frac{1}{1-y}$; which ſeries, it is manifeſt, will conſtantly converge, let the value of $\frac{1}{1-y}$ be ever ſo great; becauſe y will always be leſs than unity.

But it is to be obſerved, that this ſeries, except in its ſigns, has exactly the ſame form with that above found for the logarithm of $1+y$, and that, if both of them be added together, the ſeries $2y+\frac{2y^3}{3}+\frac{2y^5}{5}+\frac{2y^7}{7}$, &c. thence ariſing, will be more ſimple than either of them, ſince one half of the terms will be intirely deſtroyed thereby.

Since, therefore, the ſum of the logarithms of any two numbers is equal to the logarithm of the product of thoſe numbers, it is manifeſt that $2x+\frac{2x^3}{3}+\frac{2x^5}{5}$, &c. will truly expreſs the logarithm of $\frac{1+x}{1-x}$; which ſeries converges ſtill faſter than $x+\frac{x^2}{2}+\frac{x^3}{3}$, &c. not only becauſe the even powers are here deſtroyed, but becauſe x, in finding the logarithm of any given number (N) will have a leſs value.

And, in order to determine what this value of x muſt be, make $\frac{1+x}{1-x}=N$, and then x will be found $=\frac{N-1}{N+1}$; but if the quantity propoſed $\left(\frac{P}{Q}\right)$ be a fraction, inſtead of a whole number, make $\frac{1+x}{1-x}=\frac{P}{Q}$, and you will have $x=\frac{P-Q}{P+Q}$; and either of theſe values being ſubſtituted in the foregoing ſeries $2x+\frac{2x^3}{3}+\frac{2x^5}{5}$, &c. will give the hyperbolic logarithm of the number required.

Now, by finding *Neper*'s logarithm of any number, according to the foregoing method, *Briggs*'s, or the common logarithm of the ſame number, may be found as follows:

Briggs's logarithm of any number, is to *Neper*'s logarithm of the ſame number, as *Briggs*'s logarithm of 10, is to *Neper*'s logarithm of 10.

But *Briggs*'s logarithm of 10 is 1, and *Neper*'s logarithm of 10 is 2.302585093; and, therefore, if *Briggs*'s, or the common logarithm of any number, be denoted by C.L, and *Neper*'s, or the hyperbolic logarithm of the ſame number, by H.L. we ſhall have 2.302585093 : 1 :: H.L. : C.L; or $\mathrm{H.L} \times \frac{1}{2.302585093} = \mathrm{H.L} \times .4342944819 = \mathrm{C.L}$, as was required*.

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* There are, beſides theſe, many other ingenious methods, which later writers have diſcovered, for finding the logarithms of numbers, in a much eaſier way than their original inventor; but as they cannot be underſtood without a knowledge of ſome of the higher branches of the mathematics, I have thought proper to omit them, and muſt beg leave to refer the reader to thoſe works that are written expreſsly upon the ſubject.

It would, likewiſe, much exceed the limits of this compendium, to point out all the peculiar artifices that are made uſe of for conſtructing an entire table of theſe numbers; ſuch as thoſe of *Gardiner*, *Sherwin*, and others, who have treated on this ſubject.

It will be ſufficient to obſerve, that the logarithms of all the prime numbers being had, thoſe of the compoſite numbers may be found only by means of addition and ſubtraction. Thus, $L.4 = 2L.2$; $L.10 = L.2 + L.5$; $L.5 = L.10 - L.2$; $L.6 = L.2 + L.3$, and ſo on for any other of theſe numbers.

OF THE METHOD OF USING A TABLE OF LOGARITHMS.

Having explained the method of making a table of the logarithms of numbers greater than unity, the next thing to be done is, to ſhew how the logarithms of fractional quantities may be found. And, in order to this, it may be obſerved, that as we have hitherto ſuppoſed a geometric ſeries to increaſe from an unit on the right hand, ſo we may now ſuppoſe it to decreaſe from an unit towards the left; and the indices, in this caſe, being made negative, will ſtill exhibit the logarithms of the terms to which they belong.

Thus, Log. $-3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3$, *&c.*
Num. $\frac{1}{1000} \quad \frac{1}{100} \quad \frac{1}{10} \quad 1 \quad 10 \quad 100 \quad 1000$, *&c.*

Where $+1$ *is the logarithm of* 10, *and* -1 *the logarithm of* $\frac{1}{10}$; $+2$ *the logarithm of* 100, *and* -2 *the logarithm of* $\frac{1}{100}$, *&c.*

And from hence it appears, that all numbers, conſiſting of the ſame figures, whether they be integral, fractional, or mixed, will have the decimal parts of their logarithms the ſame.

Thus, the logarithm of 5874 *being* 3·7689339, *the logarithms of* $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, *&c. part of it will be be as follows:*

Num.	Logarithms.
5874	3·7689339
587·4	2·7689339
58·74	1·7689339
5·874	0·7689339
·5874	−1·7689339
·05874	−2·7689339
·005874	−3·7989339

From this it alſo appears, that the *index*, or *characteriſtic*, of any logarithm, is always one leſs than the number of figures which the natural number conſiſts of; and this index is conſtantly to be placed on the left hand of the decimal part of the logarithm.

When there are integers in the given number, the index is always affirmative; but when there are no integers, the index is negative, and is to be marked by a line drawn before it, like a negative quantity in algebra.

Thus, a number having 1, 2, 3, 4, 5, *&c. integer places.*
The index of its log. is 0, 1, 2, 3, 4, &c.

And a fraction having a digit in the place of primes, ſeconds, thirds, fourths, &c.
The index of its logarithm will be —1, —2, —3, —4, *&c.*

It may alſo be obſerved, that though the indices of fractional quantities are negative, yet the decimal parts of their logarithms are always affirmative; and all operations are performed by them, in the ſame manner as by negative and affirmative quantities in algebra.

In taking out of a table the logarithm of any number, not exceeding 10000, we have the decimal part by inſpection; and if to this the proper characteriſtic be affixed, it will give the complete logarithm required.

But if the number, whoſe logarithm is required, be above 10000, then find the logarithm of the two neareſt numbers to it, that can be found in the table, and ſay, as their difference : the difference of their logarithms : : the difference between the neareſt number and that whoſe logarithm is required : the difference of their logarithms, *nearly*; and this difference being added to, or ſubtracted from, the neareſt logarithm, according as it is greater or leſs than the required one, will give the logarithm required, *nearly*[*].

Thus, let it be required to find the logarithm of 367182.

The decimal part of 3671 *is, by the table* .5647844; *and of* 3672 *is* .5649027;

$$\therefore \begin{matrix}\textit{The}\\ \textit{log. of}\end{matrix}\left\{\begin{matrix}367100 & \textit{is} & 5.5647844\\ 367200 & \textit{is} & 5.5649027\end{matrix}\right\}$$

Their diff. 100 .0001183 *diff.*

$$\begin{matrix}\textit{Nearest } N^{o}\\ \textit{Given } N^{o}\end{matrix}\left\{\begin{matrix}367200\\ 367182\end{matrix}\right.$$

18 *diff.*

Therefore 100 : .0001183 :: 18 : .0000212,

And 5.5649027 — .0000212 = 5.5648815 = *logarithm of* 367182 *nearly.*

If the number conſiſts both of integers and fractions, or is entirely fractional, find the decimal part of the logarithm as if all its figures were integral; and this, being prefixed to the proper characteriſtic, will give the logarithm required.

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[*] This method being founded on the ſuppoſition, that the logarithms of all numbers between 367100 and 367200, increaſe, or decreaſe, equally, according to their diſtance from 367100 or 367200, is not ſtrictly true, but nearly ſo; and the greater any numbers are with reſpect to their difference, the nearer will thoſe differences be proportional. And, therefore, though this will not give the exact logarithm, yet it will be a very near approximation, and is ſufficiently exact for moſt practical purpoſes.

To find the logarithm of a proper fraction; ſubtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm ſought; which, being that of a decimal fraction, muſt always have a negative index.

And to find the logarithm of a mixed number, reduce the given number into an improper fraction; then ſubtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm ſought.

In finding the number anſwering to any given logarithm, the index, if *affirmative,* will always ſhew how many integral places the required number conſiſts of; and, if *negative,* in what place of decimals the firſt, or ſignificant figure, ſtands; ſo that if the logarithm can be found in the table, the number anſwering to it will always be had by inſpection.

But, if the logarithm cannot be exactly found in the table, find the next greater, and the next leſs, and then ſay, As the difference of theſe two logarithms : the difference of the numbers anſwering to them : : the difference between the given logarithm and the neareſt tabular logarithm : a fourth number; which added to, or ſubtracted from, the natural number anſwering to the neareſt tabular logarithm, according as that logarithm is leſs or greater than the given one, will give the number required, *nearly.*

Thus, let it be required to find the natural number anſwering to the logarithm 5.5648815.

The next leſs and greater logarithms, in the table, are

5.5647844	*The numbers*	367100
5.5649027	*anſwering*	367200
Their diff. .0001183		100 *diff.*

And $5.5649027 - 5.5648815 = .0000212$,
Therefore $.0001183 : 100 :: .0000212 : 18$ *nearly.*
Whence $367200 - 18 = 367182 =$ *number required**.

MULTIPLICATION *by* LOGARITHMS.

RULE.

Add the logarithms of the factors together, and their ſum will be the logarithm of the product required.

Obſerving to add what is to be carried from the decimal part of the logarithm to the ſum of the affirmative indices:

And that the difference between the affirmative and negative indices, is to be taken for the index to the logarithm of the product.

EXAMPLES:

1. Let the number 256 be multiplied by 4.

The log. of $256 = 2.4082400$
The log. of $4 = 0.6020600$

The product $= 1024 \ldots 3.0103000$

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* Directions, at large, for the uſing of logarithms, may be found in moſt of the common tables upon this ſubject.—*Sherwin*'s Mathematical tables, of the *Edition* 1741, or 1742, are reckoned the moſt correct and convenient, for practical purpoſes, of any now extant.

2. Let the number 8.5 be multiplied by 10.

The log. of 8.5 = 0.9294189
The log. of 10 = 1.0000000

The product = 85 .... 1.9294189

3. Let the number 46.75 be multiplied by .3275.

The log. of 46.75 = 1.6697816
The log. of .3275 = —1.5152113

The product = 15.31 .... 1.1849925

4. Multiply 3.768, 2.053, and .007693 continually together.

The log. of 3.768 = 0.5761109
The log. of 2.053 = 0.3123889
The log. of .007693 = —3.8860957

The product = .059511 ... —2.7745955

5. Multiply .5, .4, and .12, continually together.

The log. of .5 = —1.6989700
The log. of .4 = —1.6020600
The log. of .12 = —1.0791812

The product = .024 ... —2.3802112

DIVISION *by* LOGARITHMS.

RULE.

From the logarithm of the dividend ſubtract the logarithm of the diviſor, and the number agreeing to the remainder will be the quotient required.

But obſerve to change the index of the diviſor from negative to affirmative, or from affirmative to negative, and then the difference of the affirmative indices muſt be taken for the index to the logarithm of the quotient.

And, alſo, when an unit is borrowed, in the left hand place of the decimal part of the logarithm, add it to the index of the diviſor; but if it be negative ſubtract it; and let the index ariſing from thence be changed and worked with as before.

EXAMPLES:

1. Let the number 56 be divided by the number 4.

The log. of 56 = 1.7481880
The log. of 4 = 0.6020600

The quotient =14.... 1.1461280

2. Let the number 50.75 be divided by the number .25.

The log. of 50.75 = 1.7054360
The log. of .25 = −1.3979400

The quotient = 203...... 2.3074960

3. Let the number 24 be divided by the number 80.

The log. of .24 = −1.3802112
The log. of .80 = 1.9030900

The quotient .003 . . . . −3.4771212

4. Let the number .01265 be divided by the number .35.

The log. of .01265 = −2.1020905
The log. of .35 = −1.7403627

The quotient =.023 . . . . −2.3617278

INVOLUTION *by* LOGARITHMS*.

RULE.

1. Seek the logarithm of the given number in the table.

2. Multiply the logarithm, thus found, by the index of the propoſed power.

3. Find the number correſponding to the product, and it will be the power required.

Note, In multiplying a logarithm with a negative index, by any affirmative number, the product will always be negative.

But what is to be carried from the decimal part of the logarithm will always be affirmative:

---

* The rule of proportion is performed by adding the logarithms of the two laſt terms, and ſubtracting the logarithm of the firſt.

And therefore their difference will be the index of the product; and is conſtantly to be made of the ſame kind with the greater.

EXAMPLES:

1. Required the ſecond power of the number 3.874.

The log. of 3.874 = 0.5881596
The index = 2
The power = 15.01 .... 1.1763192

2. Required the third power of the number 2.768.

The log. of 2.768 = 0.4421661
The index = 3
The power = 21.21 .... 1.3264983

3. Required the third power of the number .7916.

The log. of .7916 = —1.8985058
The index = 3
The power = .4961 ... —1.6955174

4. Required the twelfth power of the number 1.539.

The log. of 1.539 = 0.1872386
The index = 12
The power = 176.6 .... 2.2468632

3. Let the number 24 be divided by the number 80.

The log. of .24 = —1.3802112
The log. of .80 = 1.9030900

The quotient .003....—3.4771212

4. Let the number .01265 be divided by the number .35.

The log. of .01265 = —2.1020905
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Note, In multiplying a logarithm with a negative index, by any affirmative number, the product will always be negative.

But what is to be carried from the decimal part of the logarithm will always be affirmative:

---

* The rule of proportion is performed by adding the loga- of the two laſt terms, and ſubtracting the logarithm of

3. What is the cube root of the number .166375?

The log. of .166375 = —1.2210881
Therefore 3) —1.2210881

The root = .55 … —1.7403627

4. What is the ſquare root of the number .08162?

The log. of .08162 = —2.9117966
Therefore 2) —2.9117966

The root = .2857 … —1.4558983

5. What is the twelfth root of the number 176.6?

The log. of 176.6 = 2.2469907
Therefore 12) 2.2469907

The root = 1.539 …. .1872492

MISCELLANEOUS QUESTIONS.

1. A perſon being aſked what o'clock it was, anſwered, that it was between 8 and 9, and that the hour and minute hands were exactly together; what was the time?

Anſ. $\overset{h.}{8} : \overset{\prime}{43} : \overset{\prime\prime}{38}\frac{2}{11}$.

2. Divide the number 50 into two ſuch parts, that $\frac{3}{4}$ of one part, added to $\frac{5}{6}$ of the other, may make 40. *Anſ.* 20 *and* 30.

3. What two numbers are thoſe, whoſe difference is 12, and their ſquares equal to each other?

Anſ. +6 *and* —6.

4. There is a certain number, conſiſting of two places, which is equal to the difference of the ſquares

3. Let the number 24 be divided by the number 80.

The log. of .24 = −1.3802112
The log. of .80 = 1.9030900

The quotient .003 . . . . −3.4771212

4. Let the number .01265 be divided by the number .35.

The log. of .01265 = −2.1020905
The log. of .35 = −1.7403627

The quotient =.023 . . . . −2.3617278

INVOLUTION *by* LOGARITHMS*.

RULE.

1. Seek the logarithm of the given number in the table.

2. Multiply the logarithm, thus found, by the index of the propoſed power.

3. Find the number correſponding to the product, and it will be the power required.

Note, In multiplying a logarithm with a negative index, by any affirmative number, the product will always be negative.

But what is to be carried from the decimal part of the logarithm will always be affirmative:

---

* The rule of proportion is performed by adding the loga[illegible]s of the two laſt terms, and ſubtracting the logarithm of

16. If money be lent, at three per cent.
To thoſe who chuſe to borrow,
In what time ſhall I be worth a pound,
If I lend a crown to-morrow?

Anſ. 46.90036 *years, allowing comp. int.*

17. Required the two leaſt nonquadrate numbers, x and y, ſuch, that x^2+y^2, and x^3+y^3 ſhall be both ſquare numbers. *Anſ.* $x=364$, *and* $y=273$.

18. There are three numbers in geometrical proportion ſuch, that if the mean be ſubtracted from the ſum of the two extremes, the remainder multiplied by the ſum of the ſaid two extremes will be 91; but if that remainder be multiplied by the ſum of all the three numbers, the product will be 133; it is required to find the three numbers by a ſimple equation. *Anſ.* 4, 6, *and* 9.

19. To determine two numbers whoſe ſum ſhall be a cube, but their product and quotients ſquares. *Anſ.* 4 *and* 4, 100 *and* 25, 900 *and* 100.

20. Required that arithmetical progreſſion whoſe number of terms is 10, ſum of the terms 185, and the ſum of the cubes of the terms 104525.

Anſ. 5, 8, 11, 14, 17, 20, 23, 26, 29, 32.

21. To divide a given number (N) into 4 ſuch parts, that if any other number (n) be added to the firſt part, deducted from the ſecond, multiplied by the third, and the fourth part divided thereby, the ſum, difference, product, and quotient, ſhall be all equal to each other.

Anſ. $\frac{Nn}{\overline{n+1}|^2}-n$, $\frac{Nn}{\overline{n+1}|^2}+n$, $\frac{N}{\overline{n+1}|^2}$ *and* $\frac{Nn\times n}{\overline{n+1}|^2}=$ [*parts required.*

22. Given $x^3y+y^3x=512500$, and $x^2y-y^2x=2500$; to find x and y. *Anſ.* $x=25$ *and* $y=20$.

23. Given $x+y+z=6$, $xy+xz+yz=11$, and $xyz=6$; to find x, y, and z.

Anſ. $x=3$, $y=1$, *and* $z=2$.

24. To find two numbers in the ratio of 5 to 7, and which being reſpectively divided by 9 and 13, ſhall leave 3 and 8 for remainders.

Anſ. 210 *and* 294.

25. To find three numbers ſuch, that $\frac{1}{2}$ the firſt, $\frac{1}{3}$ of the ſecond, and $\frac{1}{4}$ of the third, ſhall be $=62$; $\frac{1}{3}$ of the firſt, $\frac{1}{4}$ of the ſecond, and $\frac{1}{5}$ of the third $=47$; and $\frac{1}{4}$ of the firſt, $\frac{1}{5}$ of the ſecond, and $\frac{1}{6}$ of the third $=38$. *Anſ.* 24, 60, *and* 120.

26. Given $x+\frac{y}{2}=357$, $y+\frac{z}{3}=476$, $z+\frac{w}{4}=595$, and $w+\frac{x}{5}=714$; to find x, y, z, and w.

Anſ. $x=190$, $y=334$, $z=426$, and $w=676$.

27. To find four numbers x, y, z, and w, having the product of every three given; viz. $xyz=231$, $xyw=420$, $yzw=1540$, and $xzw=660$.

Anſ. $x=3$, $y=7$, $z=11$, *and* $w=20$.

28. To find four numbers in geometric proportion, whoſe ſum is 15, and the ſum of their ſquares 85. *Anſ.* 1, 2, 4, 8.

29. To find three numbers, x, y, and z, when the product of each by the ſum of the other two are given; viz. $x\times\overline{y+z}=48$, $y\times\overline{x+z}=39$, and $z\times\overline{x+y}=63$.

Anſ. $x=4$, $y=3$, *and* $z=9$.

30. What number is that, which, being any how divided, the ſquare of one part, when added to the other part, ſhall always be a ſquare number?

Anſ. 1 *only.*

31. Given $y^3+z=127$, $y^3+x=135$, and $x^3+y^3+z^3=1133$; to find x, y, and z.

Anſ. $x=10$, $y=5$, *and* $z=2$.

32. Given $x^2+xy=108$, $y^2+yz=69$, and $z^2+xz=580$; to find x, y, and z.

Anſ. $x=9$, $y=3$, *and* $z=20$.

33. Given $x+yz=384$, $y+xz=237$, and $z+xy=192$; to find x, y, z.

Anſ. $x=10$, $y=17$, *and* $z=22$.

34. To find the leaſt number, which being divided by 6, 5, 4, 3, and 2, ſhall leave the remainders 5, 4, 3, 2, and 1 reſpectively. *Anſ.* 59.

35. To find three numbers ſuch, that the ſum or difference of any two of them ſhall be ſquare numbers. *Anſ.* 1873432, 2399057, *and* 2288168.

36. To find two ſquare numbers ſuch, that their ſum may be a ſquare, and their difference a cube, and the ſide of the ſaid ſquare and cube equal to each other. *Anſ.* $\frac{783}{15625}$ *and* $\frac{414}{15625}$.

37. To determine the number of fifteens that can be made out of a common pack of 52 cards.

Anſ. 17264.

38. To find a fraction ſuch, that being taken from its reciprocal the remainder ſhall be a ſquare.

Anſ. Find ſuch a fraction as that its biquadrate being added to 4 is a ſquare, and it will anſwer the queſtion.

39. Given $x^2+xy+y^2=1087$, and $x^4+x^3y^3+y^4=4577295$; to find x and y.

Anſ. $x=21$ *and* $y=17$.

40. Given $x+y+z=78$, $x^2+y^2+z^2=2546$, and $xy-xz-yz=527$; to find x, y, and z.

Anſ. $x=41$, $y=28$, *and* $z=9$.

41. Given $x+y=152$, and $\overline{x-y}|^{\frac{2}{3}}\times\overline{x-y}|^{\frac{3}{2}}=8192$; to find x and y. *Anſ.* $x=108$, *and* $y=44$.

42. To find three numbers ſuch, that if to the ſquare of each the product of the other 2 be added, the ſums ſhall be ſquares. *Anſ.* 73, 9, 328.

43. Let the number of cards in a pack (p) be diſtributed into any number of heaps (n), by laying as many cards upon the bottom heap as are ſuffi-

cient to make up its number *q*; then by having the number of cards remaining in the dealer's hand, (*r*) and the number of heaps (*n*) given, it is required to find the ſum of all the bottom cards.

Anſ. $\overline{q+1}\times n+\overline{r-p}=$ *ſum required.*

44. To find 3 numbers ſuch, that if each be ſubtracted from the cube of their ſum, the remainders ſhall be cubes. *Anſ.* $\frac{13851}{85184}$, $\frac{19467}{85184}$, *and* $\frac{18954}{85184}$.

45. Given $x^x=123456789$ to find x. *Anſ.* $x=8.6400268$.

46. Given the cycle of the ſun 18, the golden number 8, and the Roman indiction 10; to find the year. *Anſ.* 1717.

47. To find 3 cube numbers ſuch, that their ſum ſhall be both a ſquare and a cube number; and if that ſum be ſquared it ſhall be a cube, and if it be cubed it ſhall be a ſquare.

Anſ. $\frac{x^6}{8}$, $\frac{8x^6}{27}$, $\frac{125x^6}{216}$; *where x may be any number whatever; if it be* =216 *they will be whole numbers.*

48. To find 3 numbers ſuch, that if each be added to the cube of their ſum, the ſums ſhall be cubes. *Anſ.* $\frac{23625}{157464}$, $\frac{1538}{157464}$, $\frac{18577}{157464}$.

49. With guineas and moidores, the feweſt, which way,
Three hundred and fifty-one pounds can you pay.
If paid every way 'twill admit of, what ſum
Do the pieces amount to?—my fortune's to come.

Anſ. 9 *guineas and* 253 *moidores; and* 37 *ways, which is* =12987 *l.*

50. Given $y^{z^{\frac{1}{2}}}=z^{x^{\frac{2}{3}}}=100$; to find x and z.

Anſ. $x=47.706$ *and* $z=1.42$.

51. Given $44000x^2+1=z^2$; to find x and y in whole numbers.

Anſ. $x=4048298 1221781$ *and* $z=$ $84917817811 42001$.

52. To find three whole numbers ſuch, that the exceſs of the greateſt above the middle number, ſhall be to the exceſs of the middle number above the leaſt, as 3 to 1; and alſo that the ſum of every two of theſe ſhall be a ſquare.

Anſ. 1362, 402, 82; *or* $4^n\times1362$, $4^n\times402$, *and* $4^n\times82$; *where* n *is any affirmative integer.*

53. Given $x+y=a$ (2), and $x^9+y^9=b$ (32), to find x and y by quadratics.

Anſ. $x=1.4697175$ *and* $y=.5302824$.

54. Given $x^y=500$, and $y^x=300$; to find x and y. *Anſ.* $x=4.6914$ *and* $y=5.5102$.

55. Given $xy\times\overline{x+z}|^2=300$, $xz\times\overline{y+z}|^2=1296$, and $yz\times\overline{x+y}|^2=432$; to find x, y, and z

Anſ. $x=1$, $y=3$, $z=9$.

56. Given $w^3+x+y+z=57$, $w+x^3+y+z=2763$, $w+x+y^3+z=1353$, and $w+x+y+z^3=153$; to find x, y, z, and w.

Anſ. $x=14$, $y=11$, $z=5$, *and* $w=3$.

57. Given $x+y=1750$, $xz+yv=22708$, $xv+yz=12292$, and $xzv+vzy=159252$; to find x, y, z, and v. *Anſ.* $x=1743$, $y=7$, $z=13$, *and* $v=7$.

58. Given $5x+7y+9z=93256$; to find all the different ſolutions in affirmative integers which the equation will admit of. *Anſ.* 13801148.

59. To find a ſquare number ſuch, that the ſum of all its aliquot parts ſhall be a ſquare number.

Anſ. 2401.

60. To find two ſquare numbers ſuch, that either of them, when added to its aliquot parts, ſhall make the ſame ſum. *Anſ.* 106276 *and* 165649.

61. To find three cube numbers ſuch, that their ſum may be both a ſquare and cube number.

$$\textit{Anſ.}\ 1,\ \frac{2048383}{274625},\ \textit{and}\ \frac{15252992}{274626}.$$

62. To find 4 whole numbers ſuch, that the difference of every two ſhall be a ſquare number.

Anſ. 1873432, 2288168, 2399057, *and* 6560657.

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